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Structuring mathematical knowledge and skills by means of knowledge graphs

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The aim of the study reported on in this paper was to develop, test and improve a cognitive tool which could help students structure their mathematical knowledge and skills. Mathematics teaching as an auxiliary subject in the context of secondary or tertiary education courses in other disciplines pays too little attention to the structure of the mathematical concepts presented. For the students, therefore, the network of relationships between these concepts does not become a part of their mathematical knowledge and skills, and is consequently not fully available for purposes of reasoning, proving, mathematicizing and solving problems. Knowledge graphs (KGs) can be used by students as a tool to visualize this structure of the concepts and the relations between them. The learning activity of structuring one's mathematical knowledge and skills can be supported by a model, the Mathematical Knowledge Graph Model (MKGM), which serves as a pre-structured heuristic framework. The elements of this model include a central concept, special cases of this concept, operations or actions on the concept, areas of application and properties of the concepts and operations. The present paper reports on a trial among five students of the Open University of the Netherlands (OUNL), who constructed a KG in accordance with the MKGM model. The paper focuses on the graphs produced by the students, their appreciation of the structuring activity and the relation between their graphs and test results.

1. Introduction

Mathematics as taught in secondary schools or in courses of higher education as an auxiliary subject supporting other disciplines, is often restricted to drilling techniques, mainly driven by assessment and examinations. This may lead to a lack of attention to essential aspects such as reasoning, proving, mathematicizing and solving more complex problems, for which a standard technique or an algorithm is not sufficient. This type of skill should be an important objective of any mathematics teaching that wants to educate students to cope in a mathematically competent way with mathematical problems and with those non-mathematical problems warranting a mathematical approach. In the past, this has been attempted in several ways, for instance on the basis of ideas from educational psychologists about learning mathematical concepts, investigations on problem solving and the 'realistic mathematics teaching' initiated by Freudenthal [1]. The first two options will be considered in greater detail in section 2. In realistic mathematics teaching, mathematical concepts are developed in the context of a problem situation, illustrating the role of mathematics as an applied science. In

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this approach, the learners experience the process of mathematicizing, which, in a strict sense, means constructing a mathematical model for a given real-life problem, solving the problem within this model and interpreting the mathematical solution in terms of the original problem. In more general terms, mathematicizing means rendering a problem situation accessible to treatment by mathematical means.

The approaches referred to above do not to guarantee that the learners become mathematically competent, i.e. able to use all of their acquired mathematical knowledge and skills in solving mathematical problems. Teaching practice often reduces mathematical knowledge and skills to a set of unrelated rules, which do not enhance the growth of mathematical competence. This is a real danger, not only in secondary but also in higher education. In distance teaching, for instance the mathematics courses offered by the Open University of the Netherlands, there is a real risk of students not proceeding beyond techniques and not learning the structure of concepts behind the techniques. It is, for instance, too easy for students to restrict themselves to the technical aspects of differentiation and integration, while ignoring the concepts behind these techniques, the situations in which they can be applied, and the relations they have with each other and with other sectors of mathematics. And even if the concepts are learned, they are often forgotten almost immediately after the examination. The present study was based on the point of view that learning to structure mathematical knowledge and skills should at least help students to become mathematically competent.

The main objectives of the present study were therefore to investigate how people can structure their mathematical knowledge and skills, to develop a new learning activity for this structuring process, and to develop, test and improve a tool for this learning activity.

Section 2 of the present paper provides the educational context, i.e. the main characteristics of the mathematics courses taught at the Open University of the Netherlands, the staff's general ideas about mathematical knowledge and skills and about solving mathematical problems, some educational ideas behind the use of knowledge graphs (KGs) in mathematics teaching, and the properties of a structure in relation with the three Van Hiele levels of thinking, concluding with a short description of the course on linear algebra that was used for the present study. Section 3 outlines the research project, more specifically the Mathematical Knowledge Graph Model (MKGM) and the questions underlying the project. Section 4 discusses the main results, namely, the knowledge graphs constructed by five students of the Open University of the Netherlands, their appreciation of the learning activity of structuring by means of a KG, the results they obtained on a test and the relation between the quality of the students' graphs and the test results. The fifth section draws some conclusions and discusses what can be learned from the study.

2. Educational context

The educational context of this study includes a number of aspects. One is the learning environment of distance teaching using mostly printed materials. Another aspect concerns ideas from teaching methodology about the relation between mathematical knowledge and skills on the one hand and solving mathematical problems on the other, or to be more specific, ideas about heuristics and metacognition, ideas about structures in general and structures in mathematics teaching in particular. A final aspect is the course on linear algebra which was used in the experiment reported on in the present paper. Each of these aspects is briefly discussed in this section.

2.1. *Self-supported learning*

As part of the Open University's curriculum in computer science/informatics engineering, students have to take a number of courses of mathematics, including courses on discrete mathematics, analysis (entitled 'continuous mathematics'), linear algebra and logic. These courses have been designed and developed to be done by students at home. Course materials include printed materials, computer programs and other electronic applications like e-mail and news groups. The latter are used for group work and offer students the opportunity to submit questions to fellow students and tutors. Students of informatics are supposed to possess the necessary infrastructure, including access to the Internet.

The printed materials not only present the mathematical content, but also have other teaching purposes, in particular those which would be provided by a teacher in regular teaching. Examples of such purposes include activating prior knowledge, explaining concepts in different words, giving hints for solving problems, asking questions which allow students to test their understanding, deliberately raising and clearing obstacles in order to optimize learning, etc. These functions are implemented by inserting in the printed materials so-called embedded support devices (ESDs). A study by Martens [2] showed that students appreciated these ESDs, and more importantly, that these ESDs seem to lead to better results. Examples of such ESDs include the explicit presentation of learning objectives at the start of each of the approximately 20 learning units of a course and the self-test presented at the end of each unit, allowing students to check to what extent they have achieved the objectives. There are also summaries, but these cannot replace the learning activity of structuring one's own mathematical knowledge and skills. It would seem to be virtually impossible to design an ESD that would provoke this learning activity, and stimulate students to construct, without outside assistance, an adequate mental idea of the overall structure of the mathematics they have studied.

Self-supported learning is nowadays a general tendency in all forms of teaching, not only in distance teaching, but also in secondary education. Wherever mathematics teaching fits into this tendency and explicit attention is given to structuring mathematical knowledge and skills, supporting tools are welcomed.

2.2. *Knowledge graphs in mathematics teaching*

The basic assumption in the present research project was that it is beneficial, both for the process of learning mathematics and for problem solving, to pay explicit attention to the structure of mathematical concepts. This means paying attention not only to mathematical theory, but also to a reflective attitude on the part of the students towards questions such as: what are the concepts, and which relations connect them? By forcing students to develop an overview of the concepts and their relations, it is hoped that they will experience advantages in storing and retrieving the elements of that overview. Of course, it is also hoped that they will adopt this as a permanent attitude.

Solving mathematical problems is an important aspect of mathematical competence. Polya [3], reflecting on his own mathematical research work, analysed which rules of thumb he was using. Nowadays, these rules are called heuristics. These are rules that can help one to find a solution to a problem, although there is no guarantee that a solution will be found. Heuristics are in a way the opposite of algorithms, which are rules that are certain to produce an answer to well-defined problems in a finite number of steps. Research by mainly American mathematicians and psychologists (e.g. Schoenfeld [4]) has shown that the Polya heuristics are useful in principle, but that students in specific situations do not know how actually to use them. To them it seems that they have to discover a new specific heuristic to a specific problem. It is clear that the use of heuristics in solving problems is only one part of the story. Another aspect is that of so-called metacognition. Metacognition is usually defined as the knowledge and directing, guiding, monitoring of one's own learning behaviour and that of others. Perrenet [5], a Dutch mathematics teacher and psychologist, puts the emphasis on problem solving by defining metacognition as the knowledge a problem solver has of his own problem-solving behaviour and its monitoring and regulation. Growth in metacognition can be achieved by reflection on learning processes or by looking back on a successful or even an unsuccessful problem-solving attempt. Kilpatrick [6] added to this reflection another idea that stems from mathematics: the notion of recursion. New knowledge is recursively built upon old knowledge. In Kilpatrick's view, recursion and reflection are key notions in any form of mathematics teaching aiming at mathematical competence.

For a long time, knowledge was regarded as a list of facts and skills stored in one's memory. In this view, mathematical knowledge was a list of definitions, theorems (with or without their proofs) and algorithms. The 1980s witnessed a change in these ideas about learning. Instead of Skinner's stimulus-response model, learning was from then on seen as information processing, with new knowledge being fitted into an existing knowledge structure. In the 1970s, Skemp [7] already spoke of the assimilation of new mathematical knowledge into existing knowledge. De Groot [8] pointed out that the activities of thinking and storing what is thought are implemented in chunks, referring to a network of concepts and their relations. Such networks are constructed in hierarchical layers, going from general concepts to special cases. This is even more true for mathematical knowledge. Skemp and many others have argued that mathematical knowledge is organized in what they call schemas: hierarchically organized conceptual structures that can be activated at a global level or in more detail, depending on the situational demands. People who do not use mathematics after school soon forget the mathematical knowledge they have acquired there, while still remembering that Pythagoras had something to do with $a^2 + b^2 = c^2$, as is often found. They no longer have at their disposal a schema in which a rectangular triangle has the property that the sum of the squares constructed on the arms is equal to the square constructed on the hypotenuse, and that this can be used to calculate distances or to construct a right angle using the lengths 3, 4 and 5. At best, they have an isolated memory ($a^2 + b^2 = c^2$) labelled 'Pythagoras'. The notion of schemata leads to the idea that in learning it is not the knowledge itself which is the main issue, but the insight into the structure of that knowledge; learning is then seen as a metacognitive strategy. By acquiring insight into the structure, learners also acquire insight into their own learning process, which may improve the quality of that learning process. Thus, in both teaching and learning mathematics, at least as much attention should be paid to the structure as to the individual elements of this structure.

Tall and others [9] have pointed out that the process that leads to mathematical knowledge, including concepts, is very important. Concepts are built by developing concept images, in which the emphasis is on the conceptual meaning, not on the definitions. In the present study, the emphasis was on the idea that the process of developing mathematical knowledge does not end after this process of building concepts, ending in the formulation of definitions. There should also be a place for the learner's task of building up the overall structure, in which the individual elements are given their own places.

Knowledge graphs, also known in the literature as conceptual graphs, concept maps or semantic networks, have been introduced for instance by Novak [10]. He wanted to have an external representation of the way people store information in their minds. Sowa [11] proved that his formalism, using concept maps, contains the first-order predicate logic. In the present study, a knowledge graph is a structured representation of acquired mathematical knowledge and skills. In the knowledge graph, concepts are represented by nodes (shown as rectangles) while the relations connecting concepts are represented by directed and labelled connections (shown as ovals). An example of such a labelled connection would be 'a square matrix' (concept) 'is a special case of' (relation) 'matrix' (concept). The directed and labelled connections express the meaning, at least partially. Constructing a knowledge graph forces students to reflect on and structure what they are supposed to learn.

2.3. *Structure of knowledge*

Van Hiele [12], reflecting on how pupils learn geometry, distinguished three levels of thinking. At the ground level, geometrical objects are concrete: they can be held in one's hand or drawn on a piece of paper. By manipulating objects of the ground level, for instance tiling a plane with rhombuses or drawing the diagonals of a square, a geometrical object transforms into the sum total of its properties. At the second level, a concept is identified by this sum total of properties. At the third and highest level, several objects of the second level are combined as part of a total theoretical framework. For the geometrical objects of school mathematics, this theoretical framework was Euclidian geometry, nowadays Euclidian vector space $\mathbf{R}^2$. In his recent book [13] van Hiele calls these three levels the visual, the descriptive and the theoretical level. The idea to identify an object at the descriptive level by its properties is very similar to the idea of schemata mentioned above. Van Hiele also relates his level theory to the ideas of the Gestalt psychologists about the nature of structure. According to them, structures are the mechanisms by which people obtain insight into the world around them and by which they cope with it. They achieve insight by means of structures. According to Van Hiele, insight can be established by adequate and intentional behaviour in new situations. In the context of mathematics, this is equivalent to mathematical competence. A structure is defined by the following four properties. The first is that a structure can be extended. A child, for instance, does this when it calls a lion a cat. Sometimes this extension is achieved according to rules formulated in words, but this formulation is not necessary. It is quite possible to restore a missing piece of wallpaper on the basis of its pattern without using any words. The second property of a structure is that of refinement: it enables one to take a closer look at individual parts of the whole. An example would be zooming in on the bones of a skeleton and comparing some parts with others, e.g. hand/arm versus leg/foot. The third property of a structure is the possibility to fit it into a coordinating structure. An example would be comparing a human skeleton with that of other mammals. The human skeleton becomes part of the coordinating structure of the mammalian skeleton. The fourth property of a structure relates to different structures which may be isomorphic. This is especially interesting in the context of mathematics. An example would be the straight lines through one point in space and the points in a plane. Nearly every rule of the former structure has an equivalent in the latter and vice versa.

If we succeed in incorporating these general ideas about what a structure is into the learning activity of structuring someones mathematical knowledge and skills, this will hopefully increase the effectiveness of that person's learning activity, because it fits in with the way people use structures in their daily life.

2.4. *Course on linear algebra*

This section describes the first five learning units of the OUNL's course on linear algebra (Zwaneveld, [14]), which were used in the trial with five students. The course is thus also an aspect of the context of this trial. The various topics of the five units are also characterised in terms of the Van Hiele levels.

Learning unit 1

Learning unit 1 outlines the domain the student is going to study. Matrices are introduced at the visual level as rectangular boxes of real numbers, for instance a table of geographical distances between cities. Vectors are special matrices, with a box width of 1. At the descriptive level, matrices and vectors are used in systems of linear equations (the concept of linear equations is assumed to be prior knowledge): the coefficient matrix, the augmented matrix and a solution vector. In addition to systems of linear equations, the following areas of application are presented at the visual level: geometry, graphs, input/output models in economics, growth of female cohorts in a population and transition probabilities.

Learning unit 2

This unit elaborates on the use of vectors in geometry at the descriptive level: the Euclidean spaces $\mathbf{R}^2$ and $\mathbf{R}^3$. The starting point is once again a system of linear equations: two equations with two variables which represent two lines in the plane. It is relevant to note that at this stage in the course it is not necessary to start with the axioms of the vector space. Thus, the treatment in this learning unit is at the descriptive level rather than at the theoretical level. It discusses sums of vectors, scalar multiplication, point, line and planes, norm, inproduct and angle, linear dependence or independence and the basis of 2- and 3-spaces. These topics are generalised to vectors in $\mathbf{R}^n$.

Learning unit 3

Starting once again from a system of linear equations, this unit elaborates on the concept of matrices. This is done at the descriptive level, though tending now and then towards the theoretical level. Matrix–matrix multiplication is treated as a generalization of matrix–vector multiplication, and the properties of associativity and non-commutativity are discussed. The diagonal, identity and inverse matrix are introduced by observing (at the visual) and manipulating (at the descriptive level) the multiplication of certain square matrices. But when the concept of invertibility is discussed, the student is taken to the theoretical level. The same line of treatment is followed with respect to the transpose operation and symmetrical matrices. Finally, properties involving inverting and transposing are treated at the theoretical level.

Learning unit 4

In this unit, a student learns how to solve a system of linear equations by means of Gaussian and extended Gaussian (Gauss–Jordan) elimination. The treatment of the underlying theory is at the theoretical level. This includes the three row operations of the elimination algorithms and the invertibility of these operations, which guarantees that a system of linear equations is transformed into an equivalent system. This leads on to the concepts of solvable or conflicting systems. The extended Gaussian elimination gives rise to the algorithm of inverting a square matrix. This also involves a theoretical aspect, namely, the link between the concepts of the solvability of a system and the invertibility of a square matrix.

Learning unit 5

This unit focuses on the concept of linear subspaces of $\mathbf{R}^n$. Once again, the treatment of this concept and other concepts derived from it, such as basis, dimension of a subspace and rank of matrix, takes place at the theoretical level. Two examples are the theorem about the invariance of the number of basis vectors, and the theorem that the linear span of a finite set of vectors is a linear subspace. The introduction of the concept of subspace itself, however, takes place at the visual level: lines and planes through the origin in $\mathbf{R}^n$ and $\mathbf{R}^3$. At the descriptive level, a subspace is characterized by the property that it is closed under addition and scalar multiplication. The unit ends with the theorem, including the proof that a system of linear equations has a solution if and only if the ranks of the coefficient matrix and the augmented matrix are the same.

3. Research project

The experiment with five students of the Open University of the Netherlands is part of a wider research project, involving the following stages: a survey of relevant theories and developments in the didactics of mathematics, especially in a context of self-supported learning; a study to examine the extent to which knowledge graphs constructed by experts match; the trial described in the present paper; and a study among high school students.

The use of knowledge graphs as a tool for structuring one's mathematical knowledge and skills arose from a research project at the Open University of the Netherlands which investigated the possibility of tailoring courses to meet the needs of individual students. In this project, knowledge graphs were used to visualize the elements of a course, especially as regards the question which element constitutes prior knowledge for which other element. This led to the idea of using similar graphs and the computer program developed for them as tools for the learning activity of structuring one's mathematical knowledge and skills [15]. This program, called KnowledgeGraph (KennisGraaf in Dutch), allows one to draw a

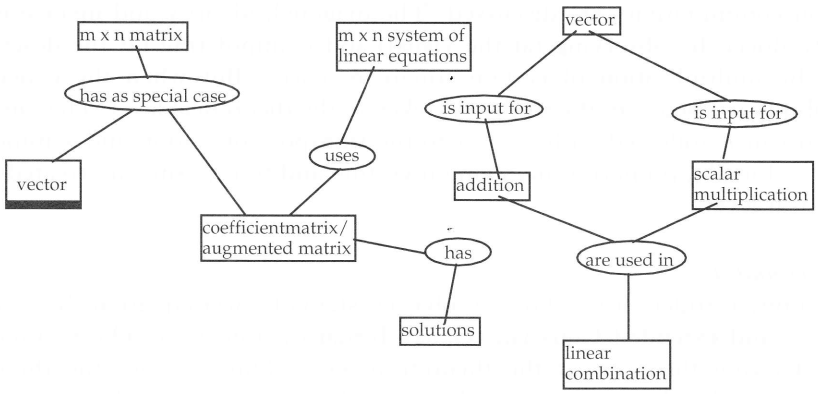

Figure 1. Left: the main KG with two entries, i.e. matrix and system of linear equations. Right: zooming in on the concept of vector (indicated by the thick black line at the bottom of the first node).

knowledge graph with the concepts showing up in rectangles, linked by directed and labelled connections with the labels in ovals. Concepts and connection labels can be moved across the screen without breaking the connections. A special feature is that it is possible to attach to each rectangle a subgraph in a new screen, giving the user the opportunity to construct a more detailed elaboration of a particular concept in a new knowledge graph. This feature is the realization of the recursive approach referred to above. Figure 1 provides an impression of this system, with the concept of vector, marked with a thick black line at the bottom, elaborated in more detail in a new KG alongside the first.

The following subsections provide a more detailed description of the student trial, including the underlying Mathematical Knowledge Graph Model, the research questions, profiles of the students and the study design.

3.1. *Mathematical Knowledge Graph Model*

At the start of the research project we had a rough idea of what a knowledge graph should be like. It had to represent the meaning of the subject matter (criterion of expressivity), and it had to make clear in one glance what it was supposed to express (criterion of convenient arrangement). This was tested in an experiment. Seven colleagues, not all of them mathematicians, but with a reasonable expertise in mathematics, including teachers and engineers, were asked to construct a knowledge graph for unit 3 of the course on linear algebra. They had an example graph at their disposal. Since this is a well-structured part of mathematics, and in view of the availability of an example graph, it was expected that the resulting knowledge graphs would be so similar that they might be regarded as representing an 'experts' graph'. Such an experts' graph could then serve as a model for students in structuring their mathematical knowledge and skills. The example provided reflected the two criteria, i.e. the labels expressed the mathematical meaning and the rectangles and labels were conveniently arranged so that the structure became clear in one glance.

It turned out, however, that the convergence was not as great as expected. Nearly all subjects recorded the concepts of the learning unit and connected them in the right way, using the labels of the example. But not all of the resulting graphs

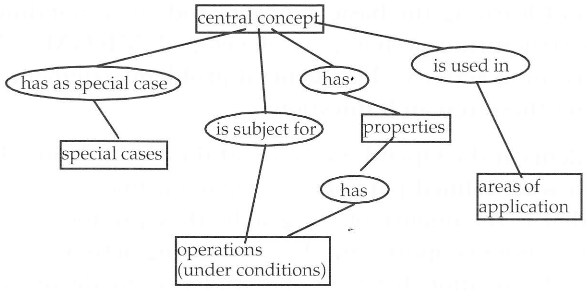

Figure 2. The Mathematical Knowledge Graph Model; each node (rectangle) allows a subgraph in a new screen to be attached, structured according to the same model.

expressed what they should express from a mathematical point of view. As regards the criterion of convenient arrangement, the resulting graphs only scored well if the constructor used the recommended option of going into greater detail by constructing a subgraph to a concept. For more details see Zwaneveld [16].

The main conclusion of this study among experts was that there was a need for a model or heuristic framework within which a student could perform his structuring activity. Therefore, the program was redesigned to provide more support. The redesigned version had to produce a knowledge graph that not only met the criteria of expressivity and convenient arrangement, but also that of significance. In addition, it had to be easy to instruct students on these criteria, so that they would be able to recognise the knowledge graph as a convenient and relevant learning tool. The criterion of significance reflects the fact that structuring knowledge and skills has its own procedure, namely, from important to less important. As a consequence, key concepts should be immediately recognizable. This approach runs parallel to that used in problem solving: a problem solver first decides what part of mathematics he thinks is most appropriate and then goes into greater detail, as was pointed out by Van Streun [17].

The redesigning process resulted in the 'Mathematical Knowledge Graph Model' (MKGM), which involves only a few central concepts. To each central concept were added four categories: special cases of the concept; operations, whether or not under specific conditions (the term operation has to be understood in the broad sense of any action the concept is involved in); properties of the concepts and/or some of the operations; and relevant areas of application (see figure 2). Any category may include an aspect, for instance a concept, that is important enough to play a role as a new central concept (in a new screen) that can be structured according to the same model. This reflects Kilpatrick's view of the recursive process of acquiring knowledge, that is, from the bottom up, while structuring is done just the other way around, from the top down.

This MKGM model was used as the basis for the learning activity in which the five students were asked to structure their mathematical knowledge and skills.

3.2. *Research questions*

The main interest of the trial was the question whether it was possible to improve students' mathematical competence, especially as regards solving mathematical problems, by introducing a new activity into the learning process, namely that of structuring their mathematical knowledge and skills, after finishing the primary process of learning the basic concepts and corresponding skills. In order to support this structuring activity, we developed MKGM, which serves as a model for this learning activity. This general problem statement was transformed into the following three research questions:

A. Are students at the Open University of the Netherlands able to construct a KG of a well-defined part of the course on linear algebra? More specifically: what is the quality of the graphs they produce?
B. Do these students appreciate the structuring activity?
C. What is the relation between the students' mode of operation and the results they achieve on a test about the subject matter?

It was hoped that the answers to question A would allow us to formulate criteria for a well-constructed KG, so a subsidiary question was: What are the criteria to assess a KG? This question is referred to as question A′.

3.3. *Students' profiles*

All ten students who enrolled in the 1996 course on linear algebra were asked to participate in the trial and five students agreed to do so. The reasons for the others not to participate were mainly that participation did not fit in with their study plans. The five participating students were ordinary students of the informatics/computer science programme for the Dutch degree of Ingenieur. Their ages ranged from 30 to 50 years. Some had a certain prior knowledge of the subject, while others had none. Two of them were employed in automation, one was a teacher at an institute of higher vocational training (not teaching mathematics), one was unemployed and one was a student of the Royal Military Academy of The Netherlands (an institute which uses some of the courses of the Open University of the Netherlands in one of its programmes). One student was female, the others were male.

3.4. *Study design*

Individual experimental sessions were held with each of the five students, each session having the same design. The session started with a short explanation of the goals and the ideas behind the research project, of the MKGM model and of the task of structuring in a KG the mathematical knowledge and skills included in the five learning units. These five learning units constitute about a quarter of the course on linear algebra. The study load of the entire course is 100 to 120 hours. The students were instructed in the use of the KnowledgeGraph computer program and allowed to practise with it for a while. The second part of the session involved constructing a knowledge graph. The students were given the choice of starting with a rough version of their knowledge graph on paper and then entering this into the KnowledgeGraph program, or to use KnowledgeGraph immediately, depending on their computer skills. In the third part of the session, the students completed a questionnaire about the learning activity and about the KnowledgeGraph program. The outcome of this questionnaire was used to answer question B. In the fourth and last part of the session, the students had to sit a short test on the subject matter. This part was included in order to find an answer to question C. During the sessions, the students were observed and asked to explain why they acted as they did. Each student's structuring activity was videotaped. The observations were described and analysed together with the graphs produced

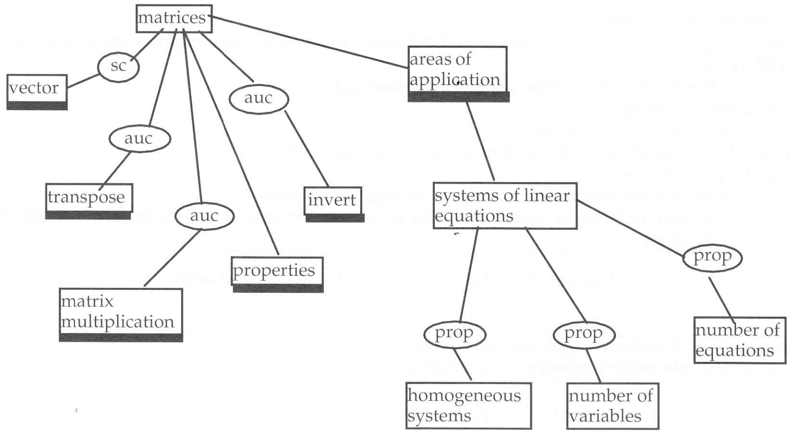

Figure 3. Main graph of the knowledge graph produced by the first student; sc = has as a special case, auc = action under condition; prop = property.

and the student's test performance, and the findings were used to answer the research questions.

This experimental design was preferred to a controlled design, because many variables cannot be controlled for students in the learning system used at the Open University of the Netherlands. For instance, there are no requirements for prior knowledge, the sequence of the individual courses is not prescribed, and students have widely different aims, some taking a full diploma programme, others only parts. These circumstances precluded any quantitative experiment.

4. Main results

This section focuses on the three main issues: the graphs produced by the students (question A), the appreciation of the activity by the students (question B) and the relation between the quality of the students' graphs and the test results (question C). Section 5 addresses the secondary question (A′) about the criteria for assessment of a student's graph.

4.1. *Question A: graphs produced by the students*

This section describes the way each student carried out the structuring task.

Student 1

The first student used the KnowledgeGraph program from the start. He systematically isolated the main topics, basing his structuring activity on the layout elements provided in the printed course materials (important concepts are marked in the left-hand margin of each page). He then fitted these topics into the model and constructed his graph. The resulting graph was almost complete, but the student had defined far more new relation labels than were used in the MKGM model. The main graph is represented in figure 3. He was quite clear as to the central concept, namely, that of matrices, and constructed separate subgraphs for several aspects of this concept: vector, transpose, matrix multiplication, invert, properties and areas of application. He considered some aspects of systems of linear equations to be so important that he included them in the main graph. These

matrix: *system of linear equations*
  vector: *inproduct, addition and scalar multiplication, dependence and independence, spanning vectors and basis*
    linear subspace: *basis, generators, linear span*
  transpose: *properties*
  matrix multiplication: *properties*
  properties: *invertibility, rank and solvability, classification of matrices*
  inverting: *properties*
  areas of application: *probability, population biology, economics*
    solving methods for systems: *Gaussian elimination, elementary operations, number of solutions*

Figure 4. KG of student 1 in shorthand.

matrix: *classification of matrices, applications, inverse*
  operations: *elementary operations on a matrix*

Figure 5. KG of student 2 in shorthand.

included homogeneous systems and some attributes, namely the number of equations and the number of variables. He called these attributes properties and connected them to the areas of application in the main graph. Other aspects, especially solving methods, were worked out in a subgraph to areas of application. The subgraph to the concept of vector also included a concept which was elaborated in a new subgraph: linear subspaces of $\mathbf{R}^n$.

Figure 4 summarizes the knowledge graph as a whole in a shorthand notation. Indentation means that a subgraph was constructed to the concept shown against the left-hand margin. For instance, the aspects of vector, transpose, matrix multiplication, properties, invert and areas of application, which were included in the main graph on the central concept of matrix, were worked out in subgraphs. The concept of linear subspace, which was included in the subgraph with vector as its central concept, was in turn worked out in a new subgraph. The descriptions of the contents of the graphs and subgraphs are shown in italics.

Student 2

It turned out that the second student had not completed the five learning units. Although he worked according to the MKGM model, he was only able to construct a graph which limited itself to some global aspects, namely the techniques of solving a system of linear equations and inverting a square matrix (the most important topics of the five learning units). He had no difficulty in deciding what was the central concept, and used KnowledgeGraph almost immediately. The central concept of his main graph was matrix, to which he constructed a subgraph about Gaussian elimination. See figure 5.

Student 3

The third student had already marked in his copy of the course material what he considered to be the important topics. At the start of the session, he wrote these topics down on a piece of paper in a short notation. Then he decided what he considered to be the central concept, hesitating between vector and matrix. He chose vector and took matrix as a special case, but later on during the session he reversed this, taking vector as a special case of matrix, but the concept of vector remained the central concept. He ended the structuring task by using KnowledgeGraph to fit the topics into the MKGM model in a consistent way. The

vector: *addition and scalar multiplication, geometry, dependence and independence, matrix-times-vector*
matrix: *matrix-times-matrix, properties of multiplication*
symmetric matrix: *special symmetric matrices*
inverse matrix: *inverting by Gaussian elimination, property*
Gaussian elimination: *elementary operations*

Figure 6. KG of student 3 in shorthand.

systems of linear equations: *solution method*
systems of linear equations: *solvability, numbers of solutions*
matrix: *matrix multiplications, properties, classification of matrices*
matrix: *inverting, properties of inverting, transpose, symmetrical matrix*
vector: *geometry*
vector: *adding and scalar multiplication*
vector: *system of vectors, dependence and independence, spanning vectors, basis, rank*
vector: *norm, inproduct, angle*
vector: *linear subspace, generators, basis, properties*
applications: *population biology, Markov matrices, Leontieff matrices, graph theory*

Figure 7. KG of student 4 in shorthand.

resulting graph was not a very thorough one. To the concept of vector in the main graph, he constructed a subgraph about vector algebra and geometry, as well as some other aspects. To the concept of matrix in the main graph he constructed a subgraph about matrix multiplication, to which he constructed three new subgraphs, about transpose, about inverse and about Gaussian elimination. See figure 6.

Student 4

The fourth student used KnowledgeGraph from the beginning. She took a long time to decide what was the central topic. In the end, she chose two main topics: matrix and system of linear equations. Next she consistently structured all the topics of the five learning units, some during a second round. The result was a very complete graph. In addition to the two main topics of matrix and system of linear equations, she included two more topics in her main KG, vector and applications, which she worked out in a subgraph. As can be seen in figure 7, the concept of matrix was given a subgraph about inverting and transposing matrices, and the concept of vector got four nested subgraphs, one about vector algebra, one about linear span, one about some geometrical aspects and one about linear subspaces.

Student 5

The fifth student also constructed an almost complete graph, using KnowledgeGraph from the beginning. Although the five central topics of the course material were also the central concepts of the graphs and subgraphs of his KG, one of these topics, methods to solve a system of linear equations, was distributed over several subgraphs in a slightly chaotic way. In the main graph there were two topics, system of linear equations and matrices. The last topic was worked out in a subgraph about matrix multiplication and areas of application, which itself had three subgraphs. One of these subgraphs, about vectors, was worked out in more detail in two subgraphs, one about the geometrical aspects (in two and three dimensions), and one about a finite system of vectors, which led to a new subgraph about linear spaces and subspaces. See figure 8.

An analysis of the videotapes of the five students' structuring activities clearly shows that all five more or less consistently used the MKGM model. Nonetheless,

systems of linear equations: *global solving*
matrix
    matrix: *multiplication, areas of application*
        matrix: *classification, transposing, inverting, rank*
        vector: *addition, scalar multiplication, dependence and independence*
            vector: *geometrical aspects*
            system of vectors: $\mathbf{R}^n$, *basis, co-ordinates*
                $\mathbf{R}^n$: *linear subspace, dimension*
        augmented matrix: *Gaussian elimination, Gauss-Jordan elimination*

Figure 8. KG of student 5 in shorthand.

all students struggled with a number of questions. These questions are discussed below, each with an example from the activities of student 4.

Is a concept so important that it should be included in a KG? If so, is it so important that it should become a central concept in a subgraph?
This question refers to the criterion of significance.

An example of this question is that student 4 at first omitted the concept of systems of linear equations, but then included it in the second round and elaborated it in a subgraph to areas of application. Regarding a further elaboration of the Gaussian elimination, she eventually decided to omit this, apparently because these algorithms can easily be looked up afterwards.

If a concept is to be included, where does it fit in?
This question, and the next two, refer to the criterion of expressivity. The student decided to include the two methods for solving systems of linear equations, Gaussian and extended Gaussian elimination, in her main KG, where she mentioned these systems for the first time. Special systems, i.e. those with exactly one, an infinite number or no solutions at all, were included in a subgraph about these systems. Extended Gaussian elimination was included a second time as a tool for inverting a square matrix.

If a concept is included, to which other concepts should it be connected?
An example is that the student at first connected Gaussian elimination to coefficient matrix and later changed this to augmented matrix.

If there is a relation between two concepts, how should this relation be labelled?
The student did not connect the concept of systems of linear equations directly to the central concept of matrix in the main graph, but in a subgraph she connected it to the concept of augmented matrix through the operations of Gaussian and extended Gaussian elimination. By doing so she apparently wanted to express the idea that these methods are operations on these matrices to provide solutions to the systems. In structuring the concepts of systems of linear equations and simultaneous systems, i.e. systems with more than one vector on the right-hand side, she remarked that the course materials presented these in a bottom-up hierarchy, but that she was using a top-down approach.

These examples illustrate that, besides other things, structuring means finding answers to the questions. Student 4's answers are not the only possible ones; in some cases, there are better answers from a mathematical point of view.

Question	Student					Mean
	1	2	3	4	5	
1.1 Able to construct a KG	2	2	2	1	1	1.6
1.2 Set of nodes available in advance	3	1	4	2	4	2.8
1.3 Set of relations available in advance	4	5	4	2	4	3.8
1.4 Expert graphs will differ	3	3	4	2	2	2.8
1.5 Constructing a KG is challenging	4	1	2	1	2	2.0
1.6 Having a KG available while studying	2	3	3	3	2	2.6
1.7 Constructing a KG is useful	3	1	2	2	2	2.0
1.8 Students will be able to construct a KG	2	3	4	2	1	2.4
1.9 Constructing a KG has a positive effect	1	1	2	2	2	1.6
1.10 A KG is helpful for solving problems	2	1	3	2	3	2.2
2.1 Interface is clear	5	4	4	5	4	4.4
2.2 Available commands are clear	1	1	2	2	2	1.6
2.3 Terminology is clear	1	1	2	1	2	1.4
2.4 Selecting with the mouse **not** clear	5	4	4	4	5	4.4
2.5 Selecting the various functions is easy	1	1	2	2	1	1.4
2.6 Control of the system is good	2	1	2	2	1	1.6
2.7 Unexpected system reactions	5	5	4	5	4	4.6
2.8 Position in the graph **not** clear	3	5	4	4	3	3.8
2.9 View of the graph is clear	3	1	4	2	3	2.6
2.10 Working at the computer is tiring	1	4	3	3	5	3.2
2.11 Adding text is useful	1	1	2	4	1	1.8
2.12 Adding a subgraph to a concept is useful	1	1	1	2	1	1.2

Table 1. Students' answers to the questionnaire; low score: agree, high score: disagree.

In addition to trying to answer these questions, there were a number of other methods which most of the students had in common. The first of these was the omission of algorithms. One example from the above descriptions is that they excluded the elementary operations of the Gaussian elimination. A second common factor is that structuring was done in a recursive way. An example is the inclusion of systems of linear equations in a second round.

By contrast, the extent to which the students used the MKGM model differed greatly from one student to the other, as is shown by the above descriptions.

4.2. *Question B: appreciation of the activity and of the program by the students*

At the end of the session, the students were asked what they thought about the structuring activity. This was done by means of a questionnaire with two sections; one containing questions about the structuring activity itself, the other questions about KnowledgeGraph. In table 1, which presents the questions in a shortened form, the 10 questions labelled 1.x belong to the first section, while the 12 questions labelled 2.y belong to the second section. The actual questionnaire presented the students with statements, on which students were asked to give their opinion on a 5-point scale.

The students' opinions on the computer program were more similar than their views on the structuring activity. They considered themselves able to construct a KG (1.1). They thought it useful for a student to construct a KG after studying the subject matter (1.7). They expected a beneficial learning effect (1.9), for instance on solving mathematical problems (1.10). They had no opinion about

Question	Type	Student 1	2	3	4	5
1	CT	10	10	10	10	10
2	CT	10	10	10	10	10
3	P	10	5	10	10	10
4	P	10	4	0	0	5
5.1	CT	10	0	0	10	10
5.2	CT	5	0	4	10	10
5.3	CT	10	0	10	10	10
5.4	P	0	0	0	10	10
5.5	P	0	0	0	10	8
Sum		65	29	44	80	83

Table 2. Scores on the test; for the assignments see the appendix

the question whether graphs constructed by diferent experts would show major differences, or about the question whether students might prefer to have a KG at their disposal while studying the subject matter (1.4 and 1.6). Some students thought it necessary to have a set of possible connection labels (1.3) and some perceived the structuring activity as a challenging one (1.5). Opinions were divided about the question whether students would, generally speaking, be able to construct a KG (1.9).

In the students' opinion, KnowledgeGraph was easy to understand (2.1–2.8), but their opinions were divided about whether it allowed them to keep an overview of the subject matter (2.9). Working on a computer screen was regarded as tiring (2.10). They found it useful that KnowledgeGraph offers the option of adding text to a concept (2.11), although nobody used this option. The option of adding a subgraph was highly appreciated (2.12).

4.3. *Question C: Relation between students' graphs and test results*

After having constructed a KG, each student sat a test on the subject matter (see Appendix). The students were assured that the test results did not influence their chances of passing the examination for the course. After the first session, one student turned out to be so exhausted from the structuring activity that he felt unable able to do the test properly, so it was decided to allow the students to do the test at home. They were asked not to consult the course materials. The test concerned the application of matrices and vectors to the growth of the female cohorts of a population. It included nine assignments. In table 2, each of the nine questions is labelled CT or P, referring to an objective classification of those problems which only rely on the preceding educational process and not on a student's abilities. A problem that can be solved by direct application of a concept (C) or a technique (T) is called a CT problem. If a problem requires one or more transformations before the solution can be found, it is called a P problem (P stands for problem). This classification was introduced by Van Streun [18]. Assignments 1 and 2 are CT problems, only requiring some calculation techniques. Assignment 3 is of the P type. The transformation required is to put the problem in the form $\boldsymbol{Ly} = \boldsymbol{x}(0)$, where y is the partition sought, while $\boldsymbol{L}$ and $\boldsymbol{x}(0)$ are given. Of course, another transformation is also possible: $\boldsymbol{y} = \boldsymbol{L}^{-1}\boldsymbol{x}(0)$. Assignment 4 is also a P problem, requiring translation in terms of systems of linear equations or matrices and vectors; the stable distribution $\boldsymbol{a}$ sought is a solution of the equation $\boldsymbol{La} = \boldsymbol{a}$. Although assignments 5.1, 5.2 and 5.3 are CT problems, some insight into the theoretical background of the concepts used is nevertheless necessary, especially for 5.2, where the students had to prove that the three given vectors are basis vectors of $\mathbf{R}^3$. It is a well-known fact that students are not familiar with proving. In the final two assignments, 5.4 and 5.5, the students had to apply the results of the preceding assignments in the conclusions, which also makes them P problems. The scores for each student are shown in table 2. The maximum score for each assignment was 10.

The five students had no problems with assignments 1, 2 and 3. On the P problem 4, only student 1 scored well, while students 2 and 5 made errors in their calculations and students 3 and 4 had no idea how to go about solving the task. Assignment 5.2 required the greatest theoretical insight of the three CT problems 5.1, 5.2 and 5.3. Only students 4 and 5 performed well here, while student 1 began well, but was unable to bring the assignment to a satisfactory conclusion. Students 1, 4 and 5 obtained the maximum scores on the other two problems, 5.1 and 5.3. On the last two P problems, only students 4 and 5 showed a good performance. Student 1 may have been too tired for these two problems. Overall, student 5 had the highest scores, while student 4 had nearly the same result. Student 1 scored very adequately, while students 2 and 3 scored poorly, with student 2 obtaining the lowest scores.

5. Conclusions and discussion

This study was designed to improve our understanding of the way people structure their mathematical knowledge and skills. This was done by following five students in their task of structuring part of the subject matter of a course on linear algebra using the MKGM model. Although five cases is not enough to allow generally valid conclusions, some things can be learned from the experiment, which will be useful for the next stages of the research project.

5.1. *Conclusion to question A: the graphs produced by the students*

A global assessment of the KGs of the five students shows that they all represent in a more or less adequate and relevant way the mathematical content of the five learning units. The second and third students constructed graphs with a restricted scope, while the third student chose the special case of vector as the central concept instead of the more general concept of matrix. The second student focused on the algorithmic aspects of solving a system of linear equations and the inversion of a square matrix. Of course, these are important issues, but they constitute only a small part of the subject matter of these five learning units. With respect to the quality of the mathematical knowledge and skills represented in a KG it can be said that the other three students constructed very satisfactory and almost complete KGs in terms of the criteria of expressivity, convenient arrangement and significance. On the whole, the graphs made by the first, fourth and fifth students were similar. Interestingly, these students were also the ones who performed well on the test.

Question A, whether students at the Open University of the Netherlands are able to construct a KG of a well-defined part of the course on linear algebra, can be answered in the affirmative. The results of students 2 and 3 show that this is only true for students who have carefully studied the subject matter and are able to look at the subject from a distance. Student 3 had carefully studied the learning units but lacked the necessary distance, while student 2 had the reverse problem: sufficient distance, but inadequate knowledge. Not all students conscientiously used the MKGM model and the recommended recursive approach. The more they did so, the better the quality of their graphs. It was found that during the structuring activity a student is indeed forced to reflect. For the fourth student, this reflection led to an improved understanding of the relation between the concepts of linear span and linear subspace.

5.2. *Question A′: criteria for graphs produced by students*

In the present trial, a student's KG was assessed from three points of view: form, mathematical content and method of construction. Form was assessed on the criteria of convenient arrangement and significance. The former means that one screen should not include more than about six or seven elements, while the latter refers to things like not including too many details for each of these elements in a screen, not emphasizing algorithms, distinguishing between major and minor concepts, and elaborating important concepts in subgraphs. The MKGM model supports these aspects.

As regards the mathematical content, we used the criterion of expressivity: the subject matter should be visualized in a mathematically correct way. The predefined relations in the MKGM model are the supporting elements in this respect. The easy part of the structuring activity is to determine the relation 'is a special case of' between concepts. For instance, the identity matrix is a special case of a diagonal matrix. And in the top-down approach to the recursive process, a student should realize that the learning materials often present the concepts in a bottom-up approach, as was noted by students 4 and 5.

The MKGM model does not guarantee that the resulting KG is mathematically correct. Although the five categories of the model, namely, central concept, special cases, operations, properties and application areas, seem clear at first sight, this is not always the case in practice. It turned out that students did not interpret the categories of special cases, operations and properties unambiguously. Some students regarded an attribute, like the dimensions of a matrix, as a property (as student 1 did), while a property is meant to be a generally valid rule, such as that matrix multiplication is associative. The attribute of dimension gives rise to a special case: the square matrix. There was a similar ambiguity with respect to the category of operations. Some students failed to mention a particular operation on a concept, but did include the special case which is the result of that operation. For instance, they might leave out the operation of inverting a square matrix, but call the inverse matrix a special case of matrix. It may be these ambiguities which cause the knowledge graphs of different students to differ in details, and they may not have recognized the operation as such. Furthermore, not all of the students immediately understood the category labelled 'operations'; some interpreted this category too narrowly. For instance, one student recognized solving an equation as an operation, but not that of establishing the solvability.

As regards the method of constructing a KG, it should be noted that this depends greatly on the way the KG is introduced to the students and the instructions they receive. The important aspects are the top-down and recursive approach. Like any other cognitive or even metacognitive skill, structuring one's mathematical knowledge and skills in this way is a protracted process that has to be practised over and over.

The main conclusion is that the criteria of form, mathematical content and method of construction are satisfactory. The MKGM model and the recursive approach provide sufficient support for the learning activity, but only on condition that students are given clear instructions on the model and the construction method and that possible ambiguities are not ignored, but are explicitly discussed.

5.3. *Question B: students' appreciation of the model*

On the whole, the five students appreciated the learning activity of structuring their mathematical knowledge and skills according to the MKGM model and considered it a useful task. They said that it offered them a new and fresh outlook on their mathematical knowledge and skills, and expected the activity to be helpful in solving mathematical problems. Apart from some minor points, they were satisfied with the supporting computer program, KnowledgeGraph.

5.4. *Question C: relation between the graphs produced by the students and their test results*

The problems of the assignment were labelled, according to the objective classification, as CT or P problems. The CT problems, involving only algorithms, led to good scores from all five students, including the two students who produced the qualitatively poorest KGs. The P type problem which needed only one transformation also resulted in good scores. But the students did not perform equally well on those CT problems involving theoretical aspects, or on the other P problems. The three students with a good KG obtained much better scores here than the two with poorer KGs. This suggests a positive relation between the scores on the test and the quality of the knowledge graphs. However, five observations are not enough to regard this as a generally valid conclusion.

5.5. *Discussion*

This section contains seven statements, each provided with some comments.

It is necessary to provide the students with the MKGM model.
It may questioned whether it is necessary to give to the students a model beforehand and force them to use it instead of leaving them completely free in the way they want to structure their knowledge and skills. An argument in favour of the latter option is that it is only the students themselves who can describe and visualize how they perceive what they have learned. A prescribed model may then put too great a limitation on the students' options. The present study opted for the first system, for two reasons. The model gives the students something to hold on to: they know how to get started and have some idea of how to look at their knowledge. Moreover, an earlier trial among seven experts, who were free to choose their own structuring method, led to knowledge graphs which were so different that it was hard to see how knowledge graphs could be used, other than as an opportunity for reflection. The possibility to check a KG constructed by the students themselves or by a teacher is then almost excluded.

Knowledge graphs produced by different students about the same subject matter are more or less the same.
Major differences in the knowledge graphs produced in the present study could be attributed to the fact that some students had not studied the subject matter adequately, that they did not pay enough attention to the underlying mathematical theory and that they lacked the necessary distance to what they had studied. Differences in the knowledge graphs may also have been due to ambiguities in the categories of the MKGM model. The most important cause of the differences in the knowledge graphs may well be differences in students' attitudes towards mathematics. This has to do with what Schoenfield calls their belief system. A student who is interested in science and technology will probably pay more attention to the areas of application than a student who is more interested in the mathematics itself.

The MKGM model and the recursive approach provide a sufficient basis to allow students to construct a KG.
Even in the limited setting of the experiment described above, this is not true. Although their instructions had been very explicit, several students included more than the recommended seven elements in one screen. Of course, this is not a very serious defect, but it points to an important phenomenon: it is hard to direct a metacognitive activity like structuring one's mathematical knowledge and skills or a mode of operation like the recursive and top-down approach by means of a model like MKGM. Students start to work in a top-down manner, but after a while they jump from the main graph to a subsubgraph and then to a subgraph, etc. Structuring in the intended sense takes a long period of learning and practice. It is not reasonable to expect that students will do well in their structuring activity after half an hour's introduction.

The KG is a useful tool for structuring mathematical knowledge and skills.
The knowledge graphs produced by the five students in this trial seem to support this statement, but, as was already shown above, the correct timing of the structuring activity in the whole learning process is also important.

The KG produced by a student provides an indication of the test result that can be expected.
The small-scale trial reported on above substantiates this statement, but five students is too small a group to allow general conclusions to be drawn.

5.6. *Epilogue*

The videotapes showed that one of the major problems for students during their structuring activity was having to answer two questions at the same time: what is the structure, and how can this structure, once found, be expressed using the MKGM model? In more general terms, one might ask whether the MKGM model might be too narrow and hamper the development of the students' own thoughts. Other aspects which have not yet been considered include the influences of prior knowledge, of experience in structuring activities and of students' learning styles.

The main conclusions of the present study are that under some conditions the construction of a knowledge graph and the underlying model are useful to support the structuring of a student's mathematical knowledge and skills. The model will gain in power once the ambiguities have been removed, for instance through better instructions, and to the extent that students practice the structuring activity.

The tool and the model were designed to support the structuring activities of students of mathematics, especially in distance teaching, because structuring is difficult to teach purely by means of printed materials. It is hoped that after having experienced the usefulness of this method, students will use it more or less automatically each time they are studying mathematics. The knowledge graph and the model could also serve another goal; it could be used in the process of designing a course, i.e. to determine the structure of the knowledge the course is supposed to teach.

The next stage of the present research project will be a trial among about 40 high school students who will be asked to construct knowledge graphs on the topics of limits and the continuity of a function. Their results will be compared with the knowledge graphs prepared by the teacher and with their performance on a test about the topic. The final stage will be a trial among the same students one year later, to find out whether the structuring activity has improved their mathematical competence and whether the KG may be considered a useful teaching tool.

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Appendix: test on the part of the linear algebra course used for the structuring activity

The female part of a population is divided into three year cohorts: juveniles, second-year adults and third-year adults. Only the second- and third-year adult females can have offspring. The second-year adults have an average of 7/3 children, while the third-year adults have an average of 4/3 children.

The average proportion of juvenile females which die is 2/3, while on average half of the class of second-year adults passes on to the third-year adult class.

This assignment concerns the development of the female section of a population with the following distribution at the start of year 0: 80 juveniles, 28 second-year adults and 12 third-year adults.

1 The vector $x(n)=[x_1(n) \quad x_2(n) \quad x_3(n)]^{\mathrm{T}}$ gives the distribution over the three year cohorts at the start of year n ($n=0,1,2,\ldots$). Here $x_1(n)$ is the number of juvenile females, $x_2(n)$ the number of second-year adults and $x_3(n)$ the number of third-year adults at the start of year n.
Explain briefly that the development of the female section of the population is described by:

$$\boldsymbol{x}(n) = \boldsymbol{L}^n\boldsymbol{x}(0), \boldsymbol{L} = \begin{bmatrix} 0 & \frac{7}{3} & \frac{4}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \text{ and } \boldsymbol{x}(0) = [80 \quad 28 \quad 12]^{\mathrm{T}}$$

2 Calculate the distribution at the start of years 1 and 2.

3 Do the same for the distribution 1 and 2 years ago.

4 A stable distribution is a vector $\boldsymbol{a} = [a_1 \quad a_2 \quad a_3]^T$ with the property that the distribution is unaltered after one year.
Does a stable distribution exist? If so, give all stable distributions.

Assignment 5 is about the question what will happen to the distribution in the long run. You can make use of the fact that matrix multiplication with a matrix $\boldsymbol{A}$ has the following two properties: $\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{Ax}+\boldsymbol{Ay}$ and for every real λ: $\boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{Ax}$.

5.1 Show that for all real α, β and γ:

$$\boldsymbol{L}\left(\alpha\begin{bmatrix}6\\2\\1\end{bmatrix}+\beta\begin{bmatrix}2\\-2\\3\end{bmatrix}+\gamma\begin{bmatrix}8\\-4\\3\end{bmatrix}\right)=\alpha\begin{bmatrix}6\\2\\1\end{bmatrix}-\tfrac{1}{3}\beta\begin{bmatrix}2\\-2\\3\end{bmatrix}-\tfrac{2}{3}\gamma\begin{bmatrix}8\\-4\\3\end{bmatrix}$$

5.2 Show that the vectors $[6 \quad 2 \quad 1]^T$, $[2 \quad 2 \quad 3]^T$ en $[8 \quad -4 \quad 3]^T$ form a basis to $\mathbf{R}^3$.

5.3 Give the vector $\boldsymbol{x}(0)$ as a linear combination of the three basis vectors from 5.2.

5.4 Calculate to 1 decimal place the distribution at the start of year 100.

5.5 Show that for any starting distribution $\boldsymbol{x}(0)$ the *proportion* of juvenile, second-year adult and third-year adult females is constant in the long run.

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