Journal of Mathematical Modelling and Application
2012, Vol. 1, No.6, 3-21
ISSN: 2178-2423

The Many Faces of the Mathematical Modeling Cycle

Jacob Perrenet
Eindhoven School of Education, Eindhoven University of Thechnology, Eindhoven
j.c.perrenet@tue.nl

Bert Zwaneveld
Ruud de Moor Institute, Open University of the Netherlands, Heerlen
Bert.Zwaneveld@ou.nl

Abstract

In literature about mathematical modeling a diversity can be seen in ways of presenting the modeling cycle. Every year, students in the Bachelor's program Applied Mathematics of the Eindhoven University of Technology, after having completed a series of mathematical modeling projects, have been prompted with a simple three-step representation of the modeling cycle. This representation consisted out of 1) problem translation into a mathematical model, 2) the solution to mathematical problem, and 3) interpretation of the solution in the context of the original problem. The students' task was to detail and complete this representation. Their representations also showed a great diversity. This diversity is investigated and compared with the representations of the students' teachers. The representations with written explanations of 77 students and 20 teachers are analyzed with respect to the presence of content aspects such as problem analysis, worlds/models/knowledge other than mathematical, verification, validation, communication and reflection at the end of the modeling process. Also form aspects such as iteration and complexity are analyzed. The results show much diversity within both groups concerning the presence or absence of aspects. Validation is present most, reflection least. Only iteration (one is passing the modeling cycle) more than once is significantly more present in the teachers' group than in the students' group. While accepting diversity as a natural phenomenon, the authors plea for incorporating all aspects mentioned into mathematical modeling education.

Keywords: mathematical modeling cycle, representations**,** higher education.

1 Introduction

From experience, supported by research (see for instance Galbraith & Stillman, 2006), it is well-known that learning mathematical modeling is a difficult task for students both in secondary and higher education. The problems that students but also teachers, are confronted with are: 1) the lack of unanimity about the essence and the vision of the modeling process; 2) the almost inherent complexity of the modeling process and, consequently, the complexity of teaching; 3) the fact that mathematical modeling is in the first place always about something, a situation and a problem arising from that situation, and that mathematics is 'only' a part of the whole process. In this article we focus on the diversity of the representations of the modeling the cycle, whereby all three problems play a role.

1.1 Representations of the mathematical modeling cycle; some examples from literature

In the research literature about teaching mathematical modeling it is agreed that the modeling process is a sort of cycle that starts and ends with a problem situation in real life or in a non-mathematical discipline, and that there is a translation of the problem into mathematical terms and a mathematical solution. However one can find a lot of modifications, extensions and improvements regarding this cycle. Examples can be found in Blomhøj & Hoff Kjeldsen (2006), Borromeo Ferri (2006) and Kaiser & Schwartz (2006). These authors often refer to the didactical representation of the modeling process by Kaiser (1995) and Blum (1996), see Figure 1. This representation is based upon cognitive psychological research on the behaviour of pupils and students working on modeling assignments (Borromeo Ferri, 2006).

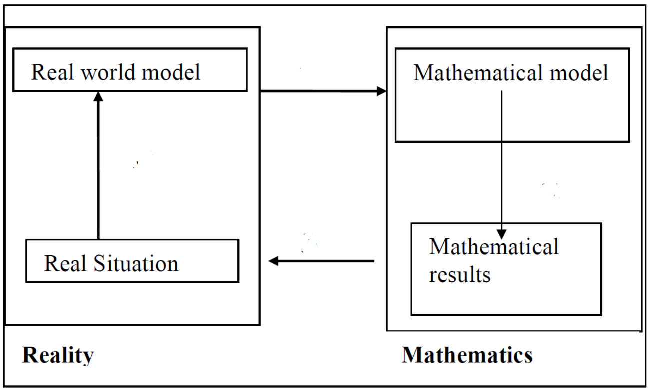

Figure 1. The Modeling cycle according to Kaiser (1995) and Blum (1996)

More recently, Blum & Leiß (2006) constructed a more detailed representation; see Figure 2.

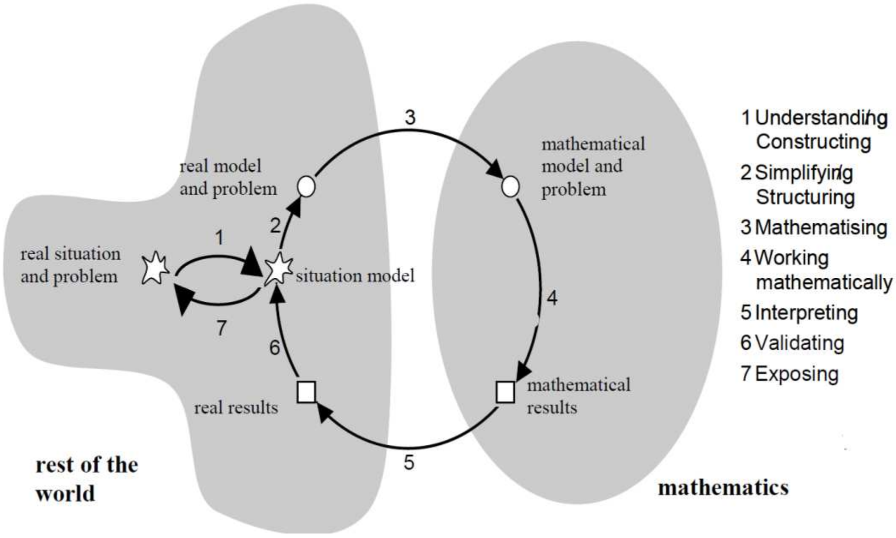

Figure 2. The modeling cycle according to Blum and Leiß (2006)

In literature, many alternative representations of modeling can be found. Various aspects are emphasized, depending on the perspective used. At the end of the 1970s Berry and Davies (1996), developed the representation of Figure 3, based upon the modeling cycle for introductory engineering education. See also Haines & Crouch (2010). We notice that for these engineering students 'reporting' has been given an explicit position in this cycle, but outside the continuing cycle.

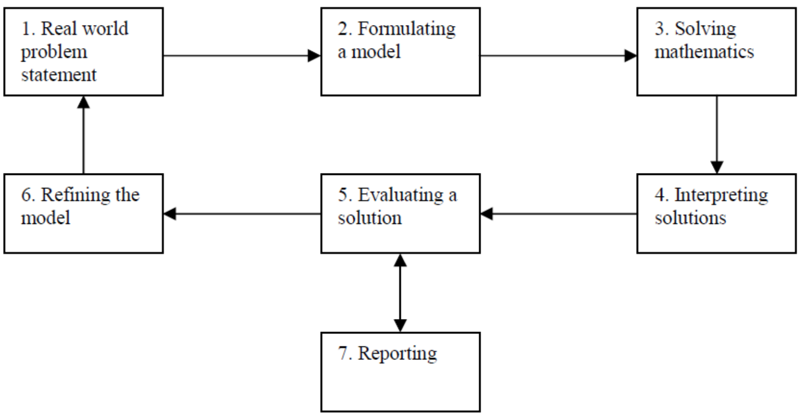

Figure 3. Modeling cycle according to Berry and Davies (2006)

The following, more recent examples indicate that the set of variations has not yet stabilized. See the representations of Carreira, Amado & Lecoq (2011), Figure 4, and Girnat & Eichler (2011), Figure 5. And see the representations with specific attention for the role of information technology by Greefrath (2011), Figure 6 (an extension of Figure 2), and by Geiger (2011), Figure 7.

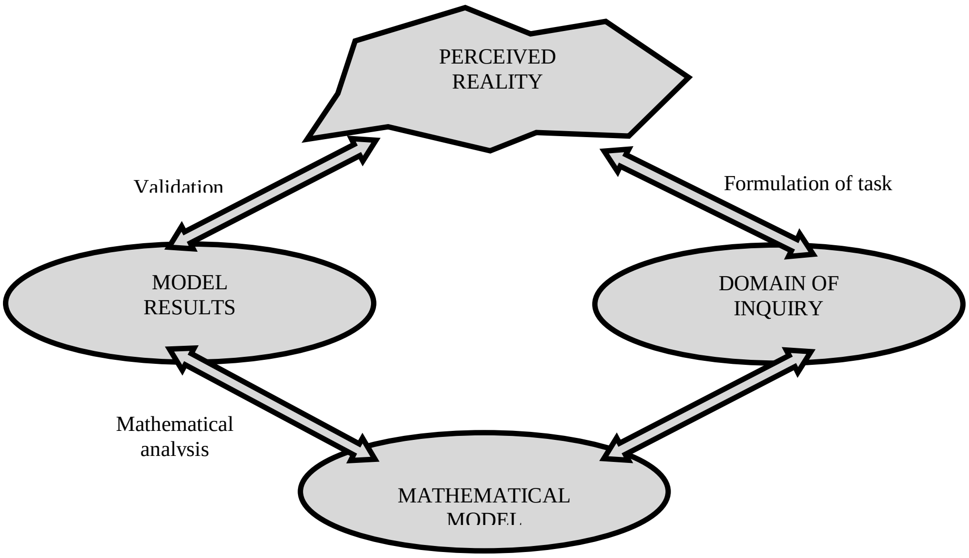

Figure 4. Modeling cycle according to Carreira et al. (2011)

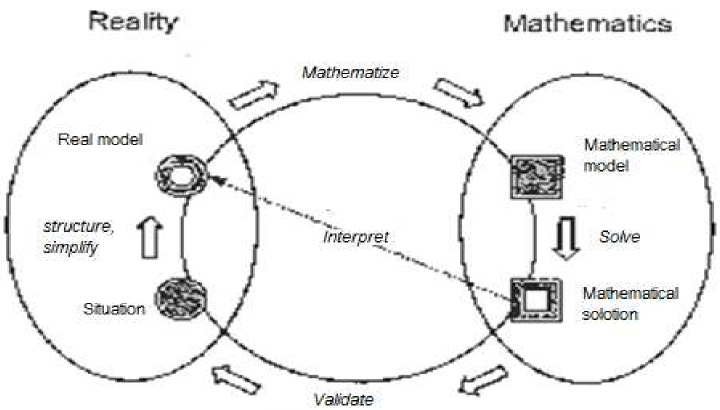

Figure 5. Modeling cycle according to Girnat and Eichler (2011)

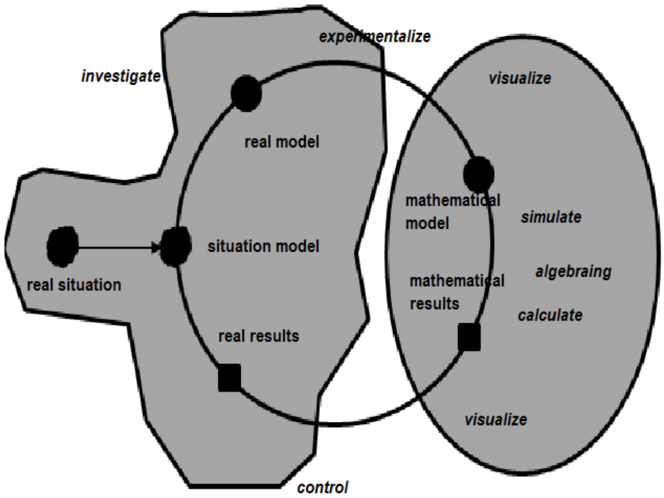

Figure 6. Modeling cycle according to Greefrath (2011)

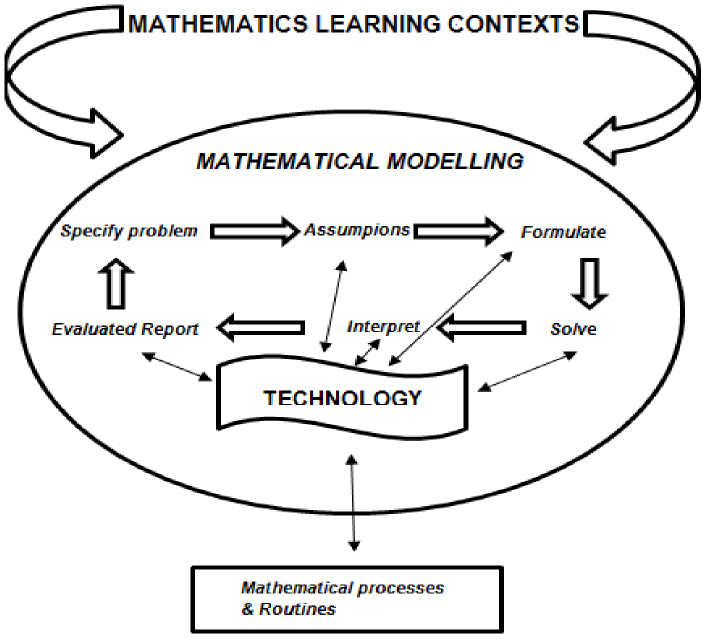

Figure 7. Modeling cycle according to Geiger (2011)

As a last example, we present a representation from our own educational context. It has been developed within the context of explaining secondary education mathematics students and mathematics freshmen about the role of mathematical modeling in the study program of Applied Mathematics of the Eindhoven University of Technology (TU/e) (Adan, Perrenet & Sterk, 2004), Figure 8. We see special attention for the phase of problem analysis with use of common sense. Also, similar to the representation of Geiger (2011) in Figure 7, the role of technology (the computer) is explicitly mentioned.

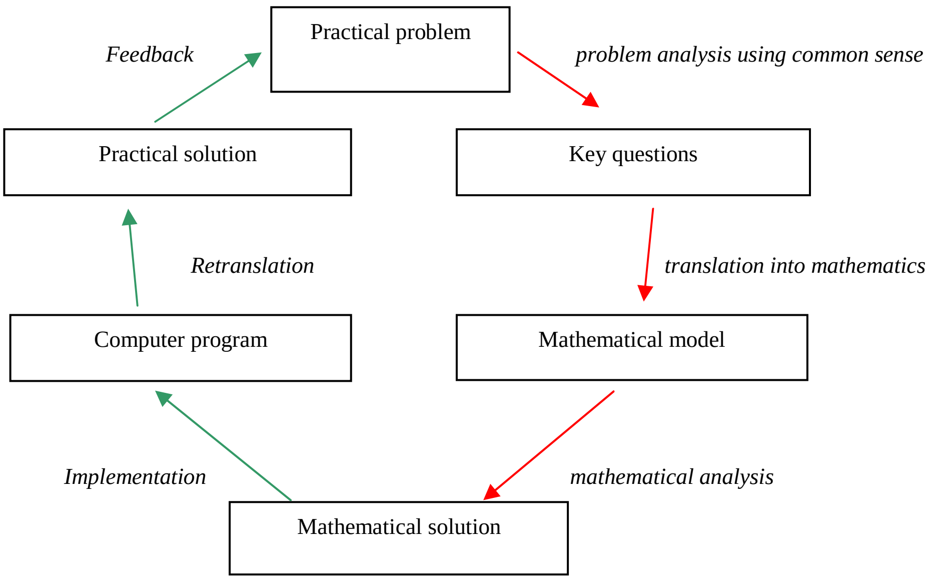

Figure 8. The modeling cycle according to Adan, Perrenet and Sterk (2004)

1.2 Diversity in representations

From this explorative review one can conclude that more or less common to all these representations (and underlying visions) is that one starts with a notion of a problem. This problem has to be translated into a mathematical model of this problem, from non-mathematical language into mathematics. Then the mathematical problem has to be solved by some kind of calculation and the mathematical solution has to be interpreted in terms of the original problem. However, there are so many diverse detailed representations of the modeling cycle that there is apparently not one overall accepted vision on modeling and the teaching of modeling (Spandaw & Zwaneveld, 2009). The following two questions illustrate this; firstly: *why should students learn to model?* See for instance Bonotto (2007) who refers to the tension between teaching the core business of mathematics (abstraction and generalization) and teaching modeling which depends critically on the characteristics of the problem situation. See also Kaiser and Maass (2007) who point out the disposition among teachers and students that the mathematics curriculum should be devoted to pure mathematics and not to handling non-mathematical situations and problems. And secondly, as a consequence of all this: *what is the best way to teach modeling?* The lack of agreement about what is the 'best' representation of the modeling cycle has at least one advantage: it stimulates the debate and serves as a topic for research (Kaiser, Blomhǿj, and Sriraman, 2006). These authors stress that the representation of course depends on the function in the teaching process. They discern six functions: retrospectively analyzing authentic mathematical modeling processes; identifying key elements in mathematical modeling competences; retrospectively analyzing students' modeling work; supporting students' modeling work and their related metacognition; as a didactical tool for planning modeling courses or projects; and as a way of defining and analyzing a curricular element in mathematics teaching.

Many researchers of mathematical modeling education are, or have been, mathematical modelers themselves. Therefore, one could assume that a diversity of representations would also be present in the community of mathematicians. It is an open question whether such a diversity would also be present in students' representations. As a teacher of a modeling course within the Applied Mathematics program at the TU/e, the first author of this article noticed a large diversity also within the population of students concluding their Bachelor program. After describing this educational context, we will come back to the research questions about this representation diversity in more detail.

1.3 Mathematical modeling in the Eindhoven program of Applied Mathematics

Mathematical modeling education at the TU/e, within the Bachelor program of Applied Mathematics, is spread over three years and consists of a series of modeling projects. See also Perrenet and Adan (2010, 2011). The goal of this program is that the students learn to apply mathematical knowledge and skills in order to solve problems posed in non-mathematical terms. The modeling courses constitute about ten percent of the Bachelor program. The students work in pairs or threes on the modeling problems. Three domains of application are involved: technology, digital communication and operational management. Every small group has its own coach, a member of faculty, and on top of that, sometimes there is an external client, someone in a real company with a real problem. Throughout the years of the program, the projects have gradually become more open, more time consuming and more complex. Also the students' dependency on the coach should decrease. The educational goal of the program is that students should *not* apply mathematical skills and knowledge that they have learned before. Rather, the students should use whatever skills and knowledge that they have or even try to master new skills and knowledge that are useful for the problem at hand. Until recently there was only a short elementary introduction to the modeling process (Figure 8) without detailed and formalized instruction.

Within the same cohort, every group gets another problem. Following are examples of problems used:

Operational management: Roundabouts

Nowadays, everywhere in the Netherlands, junctions are being replaced by roundabouts. The claim is that traffic flows faster through roundabouts. Is that true?

Digital communication: Blogs

Blogs have become a common way to present (any kind) of information on the web. Looking at the various characteristics of any recently accessible blog, would there be a systematic way to predict its future popularity (for instance, in terms of number of visits) and thus to classify a new blog as potentially popular?

Technology: Tsunami

Tsunamis are extremely high waves (caused by earthquakes) with sometimes disastrous consequences. Mathematical models play an important role in modern warning systems for tsunamis. Investigate the causes and damaging consequences of tsunamis and develop a simple model to describe the propagation of tsunamis. By making use of available geophysical data, try to use this model to predict whether tsunamis are a potential risk for the Netherlands.

Projects involve training in diverse communication skills. Connected to the series of projects is a reflection portfolio which contains a small reflection assignment after each project and a series of larger reflection assignments at the end of the third year. The first author of this article is responsible for the reflection assignments as discussed below.

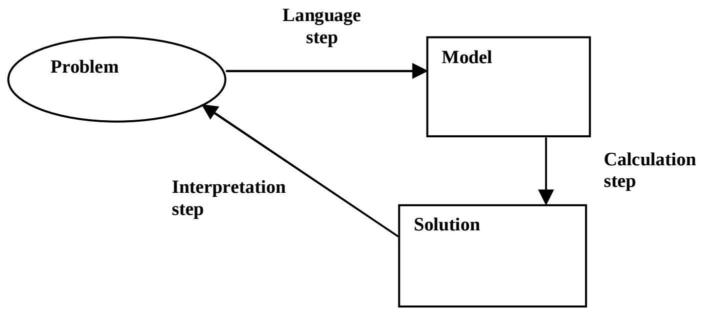

Figure 9. Elementary modeling cycle used as a stimulus for the research in this paper

The research reported in this paper is inspired by a specific reflection assignment in the third year. The students are presented with Figure 9 and the statement that this would be the essence of the mathematical modeling process. They are asked to comment and to construct a more detailed representation.

The reason for asking the students to construct (complete) a representation of such a complex process as modeling is, that it will help them to improve their understanding. From the study of Zwaneveld (1999) it appears that, in the context of mathematics education, concept mapping is a suitable tool for the visualization of cognitive structures concerning mathematical knowledge. Concept maps have been developed in the seventies of the twentieth century with the aim to visualize developing knowledge of students in the beta domain, see e.g. Novak (1977) and Sowa (1984). Concepts and their mutual relations are graphically represented, normally with the concepts placed in rectangles and the relations by means of labelled connecting arrows. Such a graphical representation maps how a student or an expert 'sees' a subject. Constructing such a graph, concept mapping, stimulates meaningful learning (Novak and Gowin, 1984). It is based upon the cognitive theories of Ausubel (1968) who, among other things, pointed at the importance of prior knowledge for the learning of new concepts. See also Novak and Cañas (2006). Many scholars have investigated the benefits of constructing concept maps by students. For example, McAleese (1998) found that the process of making knowledge explicit using knots for concepts and arrows for relations enables the student to become conscious of what he or she knows and to give it a meaning and to adapt and expand that knowledge.

1.4 Research questions

As mentioned before, the first author of this article has repeatedly observed that, at the end of the Bachelor program, there are great differences in the way mathematics students represent the modeling cycle when asked to do so. Curiosity about the degree of the differences and interest in the educational consequences of these differences were the motives for systematic research into these differences. Our perspective is focused on the fourth function of the representation of the modeling cycle, as mentioned by Kaiser et al. (2006) above: supporting students' modeling work and their related metacognition. It was decided that it would be interesting to also involve the modeling cycle representations of the teachers. In what follows, we use for short the term 'teacher' instead of 'coaching staff member'. Globally formulated, the research question is:

What diversity exists in the representations of the mathematical modeling cycle by students and teachers?

Sub questions are:

1. What differences and similarities concerning contents and form of modeling cycle representations exist between mathematics students at the end of the Bachelor program Applied Mathematics.

2. *What differences and similarities concerning contents and form of modeling cycle representations exist between mathematics teachers involved in the Bachelor program Applied Mathematic*?

3. *What differences and similarities concerning contents and form of modeling cycle representations exist between the teachers' group and the students' group involved in the Bachelor program Applied Mathematics*?

2 Methods

2.1 Respondents and stimulus task

Our respondents were students and teachers of Applied Mathematics of the TU/e. The participants consisted of 77 students and 30 teachers. The students were seven cohorts near their completion of the Bachelor program; teachers were all mathematicians connected to the study year 2009/2010 in modeling education as a coach or a client. All were presented with the elementary representation of the modeling cycle (Figure 9) and asked to give comment and expand this representation. Not all of the teachers reacted; we received 20 useful reactions (almost 70%). In contrast, the students gave a 100% response rate, as it was a compulsory assignment. The drawings and the explaining texts have been collected in order to look for systematic differences and similarities.

2.2 Selection of variables

The explorative analysis of literature (see section 1.1) suggests interesting aspects to look at, concerning the content as well as the form of the representation. As for content, we started with *problem analysis*, the presence of other *worlds* than the mathematical world, the presence of other *models* than the mathematical model, and the presence of other *knowledge* than mathematical knowledge. These four aspects can be seen as detailing the first (language) step of translating the problem into the mathematical model. For detailing the second (calculation) step, the role of the *computer* was chosen. Detailing the third (interpretation) step led to the aspects of *validation* (confrontation of the solution with what was asked for) and *communication* (with teachers), keeping in mind that their presence could also be possible at other locations in the cycle.

From our own experience, we added *verification* (confrontation of the mathematical solution with the mathematical limits and intuitions), as the counterpart of validation within the calculation step. Also, we added the aspect of reflection afterwards, *reflection* on the modeling process as a whole. Turning to the form of the representation we firstly chose *iteration*, referring to the aspect of repetence in going around the cycle as a whole. Secondly, we chose counting the numbers of nodes and edges as a measure of *complexity* of the representation.

With these aspects the two authors did a first analysis (independently) of all 20 teachers' representations and texts and a sample of 20 students' representations and texts. We discussed the outcomes with each other and consulted two experts in mathematical modeling and mathematical modeling education: Dr.Eng. Kees van Overveld and Prof.Dr.Eng. Ivo Adan, the first being a physicist and design methodologist, who has been teaching multidisciplinary modeling courses at the TU/e for many years, and the second being a mathematician and who has been coordinating the modeling education in the Bachelor program of Applied Mathematics for many years. With their help the set of variables and their definitions were refined for further analysis. The main changes were the removal of the aspect concerning the role of the *computer* and the choice for another measure of *complexity*.

The presence of the role of the *computer* appeared to be manifold: for calculation, for simulation, or as a means for finding information. Moreover, its role was often more or less implicit, leading to long discussions whether to score it at present or not. Finally, because of this ambiguity, it was decided to remove it from the variable list. Elaborating on the aspect of *complexity,* it should be noted that literature offers all kinds of measures to characterize the complexity of graphs, already for graphs with singular undirected edges (for an accessible first impression, see Orrison and Yong (2006)). Our first exploration showed that in our data the representations mostly have the form of directed graphs and often with multiple edges between nodes. Also it appeared from our data that edges and nodes could differ much in character and consequently, the representations as a whole. We noticed process schemes (representations with states and actions to represent transitions between states), communication schemes (representations with actors and streams of information) and even combinations of both. Because of this diversity, the choice for a manageable complexity measure was difficult. In consultancy with both experts mentioned before, we finally chose for another scoring method, first looking at the complexity of every node and then using the maximum of these local complexities as a measure of the complexity of the representation as a whole. The chosen measure also appeared to suit those cases where no representation was present but only explanatory text data. We will explain this measure of complexity further in the next sextion, along with definitions and scoring rules for the other variables.

At this point we want to emphasize that our way of looking at the data was with descriptive perspective. We were not judging the delivered representations, as in principle, for students at the end of the Bachelor program and certainly for teachers, the representations delivered should be correct by definition.

2.3 Operationalization of variables and analysis

- *Problem Analysis:*Is it mentioned (score 1) or not (score 0) in the representation or explanatory text that in the beginning of the process the problem is analyzed? Here it does not concern mathematical assumptions, rather it does concern a non-mathematical analysis of the problem, such as the answer to the question: *what is really relevant?* Or: *what is really the problem?*

- *Worlds:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that the modeling cycle not only takes place in the mathematical world, but also in several other worlds? And if so, which ones?

-*Models:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that in the modeling cycle several types of models (other models than the mathematical model only) are used? If so, which ones?

- *Knowledge:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that other than only mathematical knowledge is used and, more specifically, domain specific knowledge? If so, what kind?

- *Verification:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that the mathematical model has to be tested and adapted against mathematical logic and consistency?

- *Validation:*Is mentioned (score 1) or not (score 0) in the representation or in the explanatory text that a mathematical model has to be tested and adapted against the requirements of practice?

-*Communication:*Is mutual interaction with the coach or client mentioned (score 1) or not (score 0) in the representation or explanatory text?

- *Reflection:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that after finding a satisfying solution one should look back at the process as a whole, reflecting on what could be used or improved for the next time?

-*Iteration:*Is it mentioned (score 1) or not (score 0) in the representation or in the explanatory text that generally it is necessary to go through the modeling cycle more than once?

- *Complexity:*For every node of a representation we counted the number of incoming and outgoing edges (relations); the so-called local complexity of a node. Next, we computed the maximal local complexity for every representation. Since all other variables were measured in a binary way, it was decided to dichotomize this one as well. Representations with a maximal local complexity above the median of all maximal local complexities were categorized as representations with a 'high degree of complexity' (score 1); the other representations were categorized as representations with a 'low degree of complexity' (score 0).

In the analysis only the representation delivered was always used at first and after that the clarifying text (if present). In some cases no representation had been constructed, but only described in relation to the stimulus representation. In those cases, we constructed a representation based upon the text. During the analysis, the other aspects that also caught the eye have been registered.

In order to ensure the reliability of our method, firstly, all data has been scored independently by both researchers. Secondly, all scores have been compared and discussed. In the great majority of cases (90 %) agreement in scores existed without discussion; for the remaining 10% only minimal discussion was needed to reach consensus. In order to further ensure the validity concerning our selection of variables, we discussed it afterwards with a sub group of fifteen teachers involved in the modeling projects. They agreed that the variables used in the study were the relevant ones (except for *reflection*).

3 Results

3.1 Examples of diversity

To give an impression of the degree of diversity, we first give a series of examples from the students' group as well as from the teachers' group. Most examples, because of the Dutch language or because they were delivered handwritten had to be edited a little for reasons of readability. Such editing only concerned the clarity of the representation, never the content or the form of the representation.

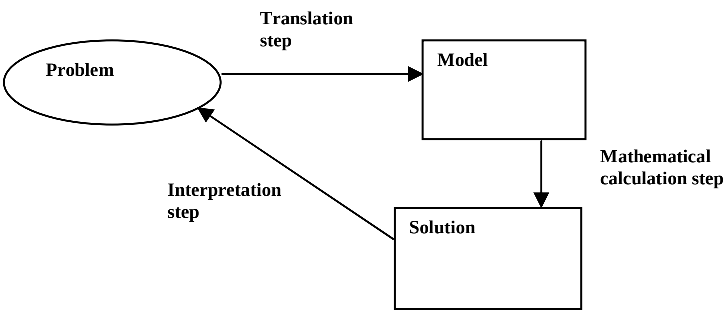

Figure 10. Example of the simplest teacher representation

The simplest teachers' representation is almost the same as the representation of Figure 9, the only difference being that 'language step' has been replaced by 'translation step' and 'calculation' by 'mathematical calculation'. See Figure 10.

A complex teacher representation can be found in Figure 11, with a maximal local complexity of 6 (at the node 'model'). In this example we see validation and verification clearly present, approximative and simplified model are specific types of models, and refinement is inherent to problem analysis and scored accordingly.

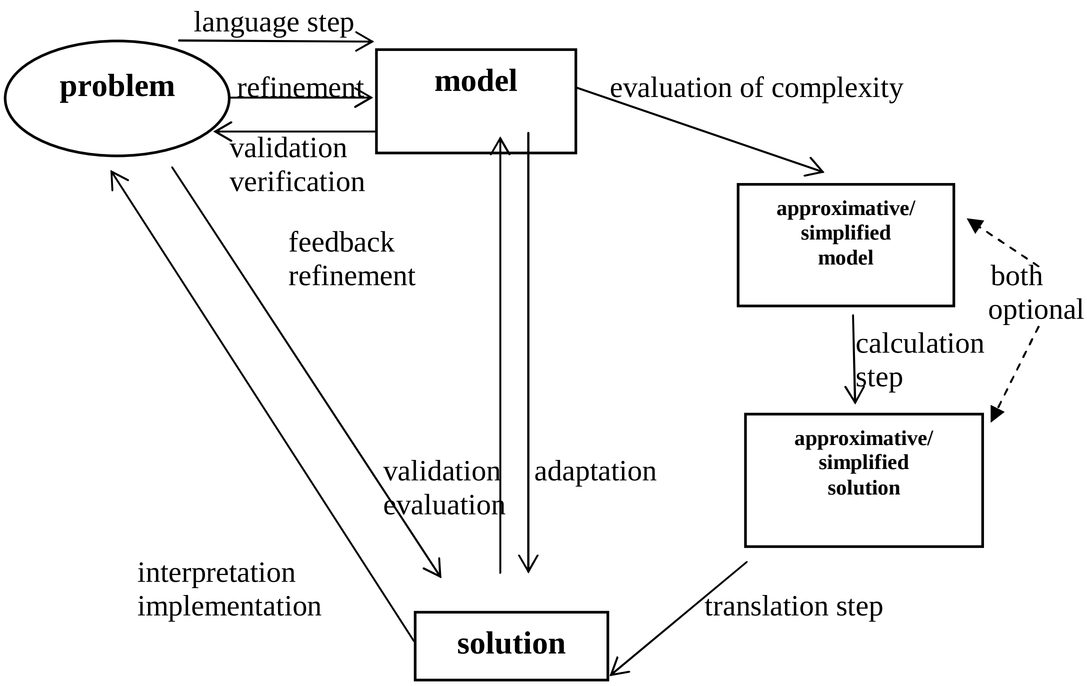

Figure 11. Example of a complex teacher representation

The simplest student representation is the one in Figure 12. We can notice 'Problem Analysis' and 'Quantities and Relations' as a specific mathematical model.

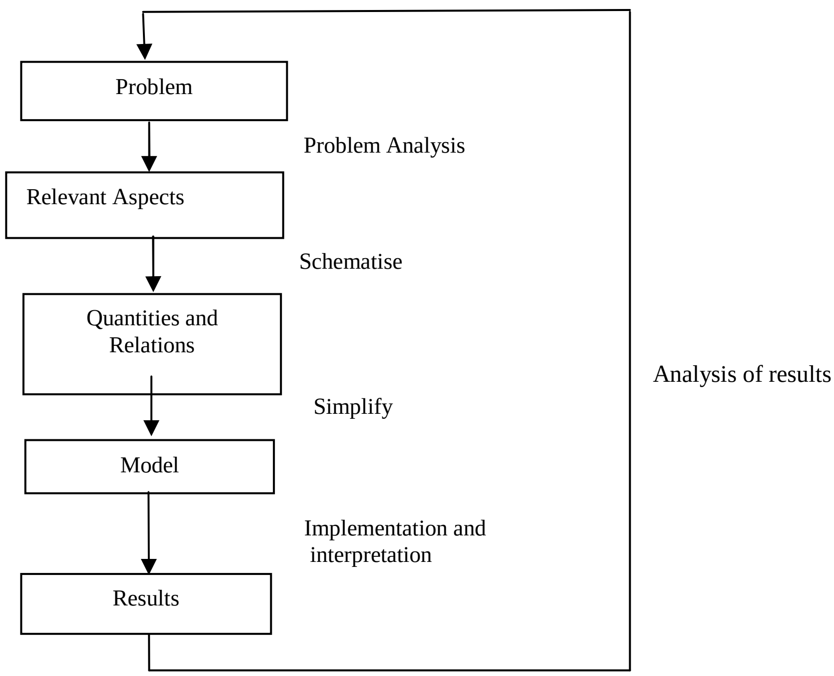

Figure 12. Example of the simplest student representation

In Figure 13 a student representation is shown wherein the client plays a central role, making this representation scores at communication (not shown here are the student's explanations of the figures 1 to 5 in his representation).

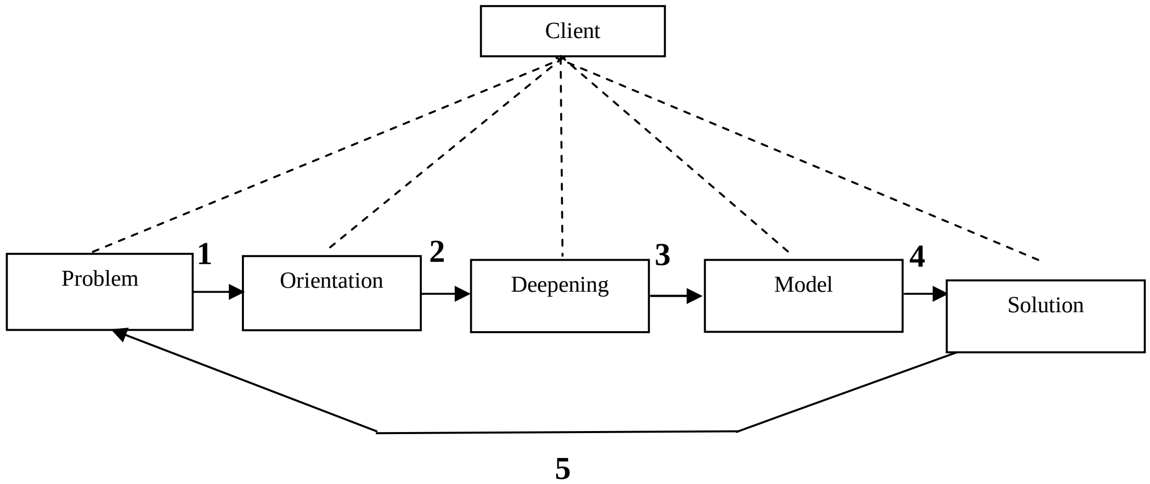

Figure 13. Example of a student representation with an explicit role for the client

In Figure 14 a student representation example is shown with distinction between the real world and the mathematical world, which therefore scores on the variable worlds.

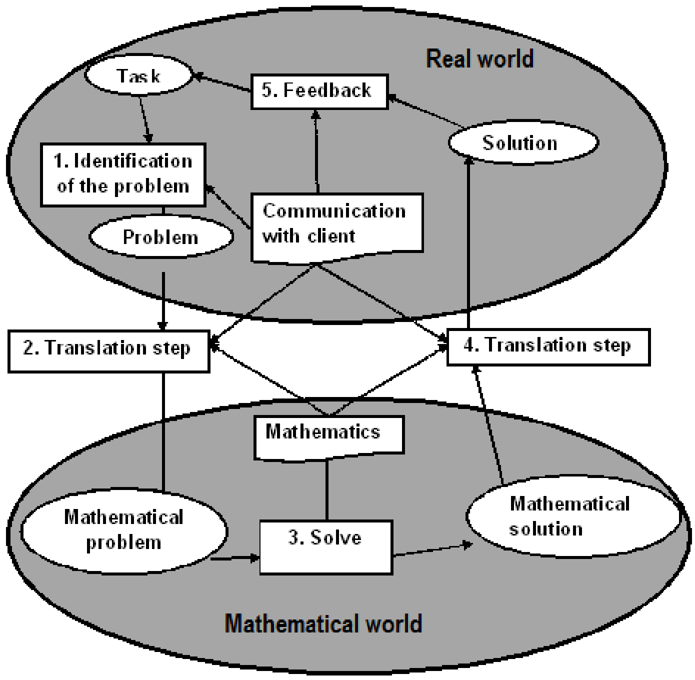

Figure 14. Example of a complex student representation with distinction between the real world and the mathematical world

Especially some teachers only delivered a text without a drawing. We give two examples; the first one contains a lot of text (we summarized it); the second one contains only a little text.

Example of a reaction (from a teacher) without a drawing but with a lot of text:

1. *Informal description of the problem*
2. *Mathematization*
3. *Hierarchy of important – unimportant effects and relations*

4. *Analysis of potential models*
   – *lower bound and upper bound within an ordered set*
   – *simplest models*
   – *small parameters/asymptotics/perturbation methods*
5. *Definition of the Minimal Model*
   – *simplicity against requirements*
6. *Continuing interaction with reality*
   – *if necessary more or less steps back*
7. *Possibly multiple cascades of models*
   – *partially ordered*

Example of a teacher's reaction without a representation and with little text: *Few thoughts arise and I am OK with the triangle.*

After this first impression of the diversity we will give an overview of the results.

3.2 Frequencies in both groups and associations between variables

Table 1 shows an overview of the frequency percentages in both groups. Regardinghe result of the aspect of complexity we can tell that some representation were so complex that we could not compute the precise maximal local complexity. We scored these as 'high' (score 1). The median value of all maximal local complexities appeared to be equal to 4.

We see in Table 1 that only at the aspect of *iteration* there is a clearly significant difference between both groups: iteration is more often present in the teachers' representations. In both groups *validation* scores more than 50% and *knowledge* and *reflection* less than 20%.

aspect	% presence at teachers (N=20)	% presence at students (N=77)
problem analysis	40	60
worlds	25	16
models	35	20
knowledge	15	10
verification	30	47
validation	65	81
communication	25	31
reflection	5	1
iteration	70**	39**
high complexity	50	53

** = difference significant at 0.05 (t-test, two-sided)

Table 1. Frequency percentages per aspect in the teachers' group and the students' group

To explore patterns, we investigated associations between variables within the groups by use of the phi coëfficiënt (Field, 2009, p. 791), which measures the degree of association between two dichotomous variables (with only values 1 and 0). For the following pairs of variables in the teachers' goup, φ was at least 0.4 at a level of significance of .05 or lower: *communication* and *problem analysis* (φ = 0.471 at significance level .035), *worlds* and *knowledge* (φ = 0.467 at significance level .037), *worlds* and *problem analysis* (φ = 0.471 at significance level .035), *problem analysis* and *validation* (φ = 0.599 at significance level .007), and *iteration* and *validation* (φ = 0.435 at significance level 0.052). However, the teachers' group is too small for further analysis into clusters. In the students' group, only some association exists between *complexity* and *verification* (φ = 0.309 at significance level .007).

3.3 Qualitative analysis of the use of terms

For the aspect of *knowledge* (other knowledge than mathematical knowledge) the terms used fell into a few categories such as literature of the domain, common knowledge and knowledge of physics. The aspects of *worlds* and *models* revealed much more diversity in the use of terms, therefore further (qualitative) analysis was performed. *Worlds* refers to whether modeling not only takes place in the mathematical world, but also in one or more other worlds; *models* refers to whether in modeling other types of models than mathematical models are used.

In Table 2 we give an overview of terms used for other worlds than the mathematical world and the frequency of occurrence (between brackets, if greater than 1). The majority of the teachers (75%, 15 out of 20, Table 1) and the majority of the students (84%, 65 out of 77, Table 1) do *not* refer to other worlds. In both groups a minority uses other terms indeed. Most of them use one other term, whereas some use several other terms. Only some other terms are used by several students and/or teachers: reality, practice and real world. Mostly, reality is mentioned as another world, but sometimes also the non-mathematical outer world or the inner world is denoted (in the teachers' group as well as in the students' group).

Students	*Teachers*
reality (4)	real world (2)
practice (3)	non-mathematical world
original world	physical world
real world	playground with attributes (e.g. of an astronomer or a plumber)
genuine world	conceptual world
non-mathematical side	'world in-between' (unlabeled)
world where the problem takes place	
perceptions of the problem situation	

Table 2. Frequency of terms for other worlds than the mathematical world

In Table 3 we give an overview of terms used for other models than the mathematical model and the frequency of occurrence (between brackets, if greater than 1). We did not make a separate list for terms like 'model' (when mathematical model is meant) and 'sub model' (when the mathematical model of a sub problem is meant). The majority the teachers (65%, 13 out of 20, Table 1) and the majority of students (80%, 62 out of 77, Table 1) do *not* refer to other models. In both groups a minority uses other terms indeed. Most of them use one other term, some use several other terms. Only some other terms are used by several students and/or teachers: simplified model, stochastic model. Mostly general terms, such as simplified model or possible model, are used; sometimes terms have a specific mathematical background, such as a stochastic model; sometimes the background is another domain, such as a physical model.

Students	*Teachers*
simple model	simplified model
simplified models (4)	simplest model
analyzable model	approaching model
computable model	stochastic model (2)
manageable model	metaphor
unusable model	first principle model
uncomputable model	empirical data model
adapted model	right model
frozen model	possible model
conceptual model	ordered set of potential models
mental model	more complete, but less transparant models
concept model	minimal model = The Model
final model	deterministic model
specified problem	continuous model
head model	discretisized model
intuitive model	computable model
physical model	minimal physical model
model (if distinct from mathematical model)	detailed model
scheme with quantities and relations	model versions
extended model	

Table 3. Frequency of terms for other models than a mathematical model

3.4 Miscelleneous

Finally, the following other interesting aspects caught the eye in individual cases during the analysis.

- project approach: mentioning that time and money are relevant
- mixing up verification/validation: in some cases (also in the teachers' group) the term 'verification' was used to refer to validation (we scored these cases as 'validation')
- decision nodes (see Figure 15 for an example)
- parallel processing (see Figure 16 for an example)

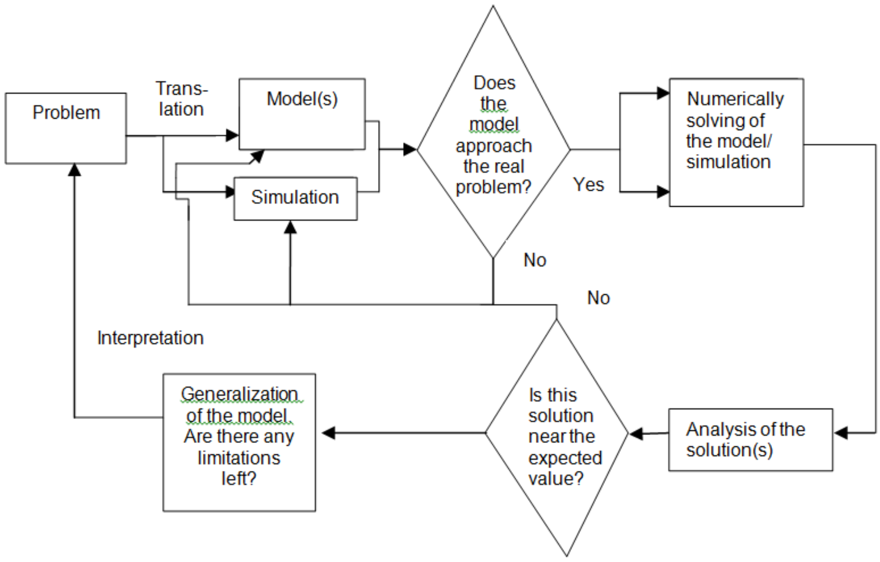

Figure 15. Example of a student representation with decision nodes

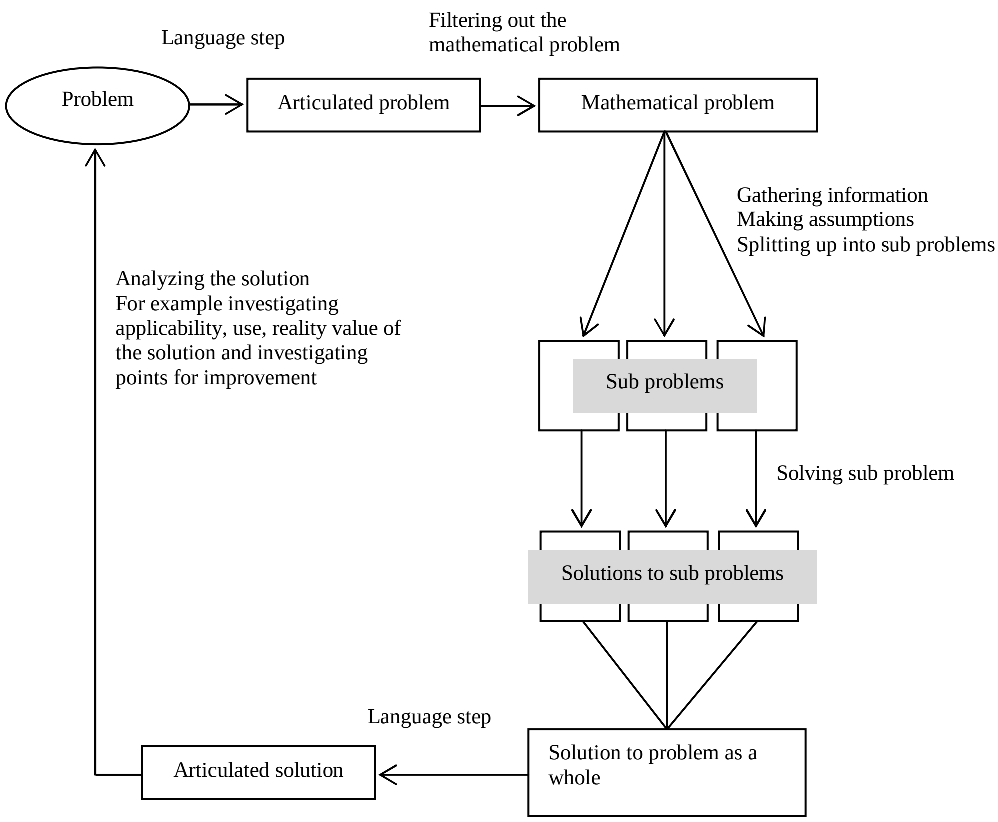

Figure 16. Example of a student representation with parallel sub processes

4 Conclusions and discussion

From our analysis of the data of teachers and students it indeed appears that there is a large diversity in the representation of the modeling cycle, from a marginal extension of the sober cycle until rather complex representations. This is true for the teachers as well as for the students. The occurrence of *problem analysis* and *validation* scores in the top three in both groups; the reference to other *knowledge* than mathematical knowledge and *reflection* scores in the bottom three in both groups.

Teachers use the term iteration significantly more often than students. A possible explanation is that the students may sometimes have to solve problems where going through the cycle once is enough or it could be that there is no time left to go through the cycle once more.

In the teachers' group the strongest association between aspects in representation is between *problem analysis* and *validation*. An explanation is that in problem analysis what is essential in the problem, is investigated. This logically asks for a validated connection between solutions and these essential elements.

Although it was not the objective to evaluate the representations on correctness, the fact that even in the teachers' group, validation and verification were confused was remarkable.

We can distinguish three factors that could explain the observed diversity. 1) From a constructivistic perspective (Cobb, Yackel, & Wood., 1992) of mathematical knowledge, representational diversity is to be expected by definition. 2) Mathematical modeling is not the same in various mathematical domains. Not only were the teachers that took part in our investigation specialists within a domain, but also the students during the conclusion of their BSC program had already chosen a mathematical specialization and had some specific knowledge of a unique sub domain. 3) Until recently, the mathematical modeling education track in the Applied Mathematics program in Eindhoven, comprised very little modeling theory for all students, but much guidance by a unique series of coaches and clients in modeling projects with unique content. For the teachers' group, a fourth factor could be thought of, namely, that some teachers answered the question more seriously than others. In the students' group, that could not be the case, as it was a compulsory assignment for them.

A critique of our method could be that the elicited representation may not mirror the real modeling behavior of students in practice. Close observation of students and the comparison of behavior with given representations would result in interesting questions for further research. Another critical remark could be that starting from scratch, instead of starting from the elementary three-step representation, would have been an even better way to measure diversity. We agree that possible diversity would have been greater, but that would support our main finding. Our result, concerning the difference of presence of iteration, was not prompted by the three-step elemenatry representation. From all our the operationalizations, the most freedom and therefore the hardest choice was at the aspect of *complexity*. We are convinced that our choice was a rational one, however we cannot exclude that other choices with possibly somewhat different results are thinkable.

Would we have the courage to generalize our results on diversity to other contexts of mathematics education? We think that an important factor would be the diversity in the theoretical and the practical experience of the modellers. With extensive explicit instruction of the modeling cycle, with more closed assignments and similar assignments for all students, representation diversity would probably decrease. However, using the constructivism argument the (first explanation factor mentioned above) we expect that even at the secondary level and even under conditions with less freedom, some diversity can be expected. Blum and Borromeo Ferri (2009, p. 48), referring to Borromeo Ferri (2007), reported that secondary school mathematical modellers used the steps of (Blum and Borromeo Ferri's) modeling cycle unsystematically. Could it not be that students used their own diverse cycle systematically?

Looking back at our investigation, we realize that we started with a descriptive perspective. Students at the end of the Bachelor program and certainly their teachers are expert modellers, so their representation of the modeling cycle is right by definition. Seeing our results concerning the mix-up by some students and even by some teachers of *validation* and *verification*, triggered a change to the prescriptive perspective. We will now answer the question: *what aspects of the modeling cycle should be present in teaching modeling?*

Of course, we require the aspects of the elementary three-step cycle presented before (Figure 9). Looking back we now prefer slightly different terms, leading to: *problem situation*, *mathematizing*, *mathematical model*, *solving*, *mathematical solution*, and *interpreting*.

This study lead us to the following extra aspects:

- *Problem Analysis:*In the beginning of the process the problem is analyzed, looking for answers to such questions as: 'What is really relevant?' Or: 'What is really the problem?'

- *Worlds, Models,* and *Knowledge:*This cluster of aspects refers to the fact that mathematical modeling is much more than modeling alone. The modeller does not work in the mathematical world only: problems come from other domains with relevant non-mathematical knowledge and relevant non-mathematical models. A specific non-mathematical model is the result of the *problem analysis* which could be called the *conceptual model,* as problem analysis is in fact *conceptualizing* the *problem situation.*

- *Verification:*The mathematical model and the solution have to be tested and adapted against mathematical logic and consistency.

- *Validation:*The mathematical model and the solution have to be tested and adapted against the requirements of practice.

- *Communication:*Mutual interaction with the coach or client (problem-owner) is necessary.

- *Iteration:*Students should receive problems that are complex enough to realize that generally it is necessary to go through the modeling cycle more than once.

- *Reflection:*Although hardly mentioned by the modellers of our population, we emphasize that mathematical modeling – just as problem solving (see, e.g. Schoenfeld, 1985, 1992) – cannot do without metacognitive activity. Reflection, especially afterwards, should not be forgotten at the moment that students, teachers and clients are pleased when an acceptable solution has been found for the problem at hand. Answering questions such as: *could the methods used be applied in other contexts*? *could the models used be applied to other modeling problems*? *what improvements were necessary after verification and validation and why*? would strengthen the capacities of the mathematical modeller for the future.

Finally, we show in Figure 17 an example of a representation of the modelling cycle with all these aspects.

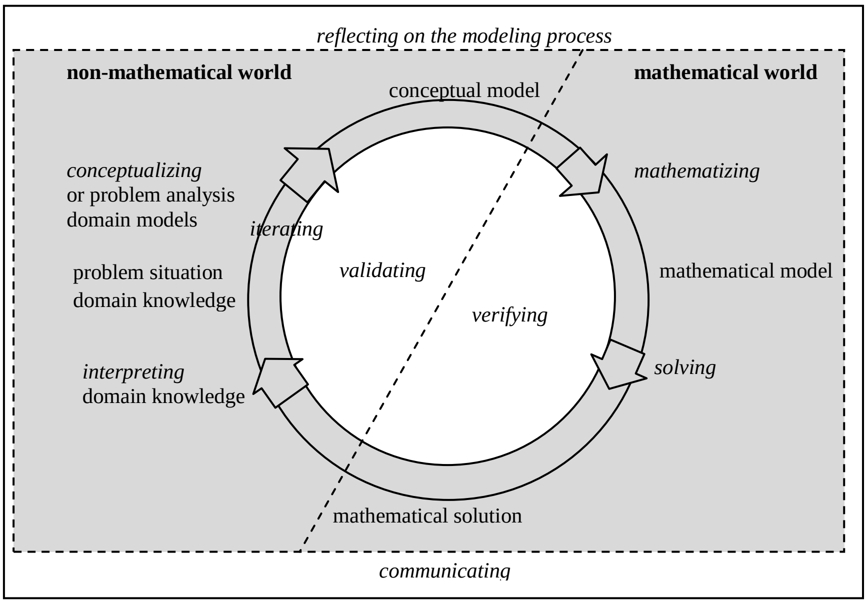

Figure 17. Modelling cycle with all aspects found in our study

Acknowledgements

We are grateful to the (former) coordinator of mathematical modeling education in Applied Mathematics, Ivo Adan, for his mediation between us and the modeling teachers. We thank Ivo Adan and especially Kees van Overveld for discussing methods of analysis of the modeling cycle representations. Finally, we are grateful to Kees van Overveld for giving feedback on an earlier version of this paper.

References

Adan, I., Perrenet, J., & Sterk, H. (2004). De Kracht van Wiskundig Modelleren. Eindhoven University of Technology: Eindhoven. [In Dutch].

Ausubel, D. (1968). *Educational Psychology: A Cognitive View*. New York: Holt, Rinehart & Winston.

Berry, J. & Davies, A. (1996). Written reports. In C.R. Haines and S. Dunthorne, (Eds.) *Mathematics Learning and Assessment: Sharing Innovative Practices*, 3.3 - 3.11. London: Arnold.

Bonotto, C. (2007). How to replace word problems with activities of realistic mathematical modeling. In W. Blum, et al. (Eds.), *Modeling and Applications in Mathematics Education. The 14th ICMI Study* (pp 405 – 408). New York: Springer.

Blomhøj, M. & Hoff Kjeldsen, T. (2006), Teaching mathematical modeling through project work. *ZDM, The International Journal on Mathematics Education*, 38(2), 163 – 177.

Blum, W. (1996). Anwendungsbezüge im Mathematik-unterricht – Trends und Perspektiven. In Kadunz, G. et al. (Eds.), *Trends und Perspektiven. Schriftenreihe Didaktik der Mathematik*, Vol. 23. Wien: Hölder-Pichler- Tempsky, 15 – 38. [in German]

Blum, W. & Leiß, D. (2006). How do students and teachers deal with modeling problems? In C. Haines, P. Galbraith, W. Blum and S.Khan (Eds.) *Mathematical Modeling (ICTMA12): Education, Engineering and Economics* (pp 222 – 231). Chichester: Horwood Publishing.

Blum, W. & Borromeo Ferri, R. (2009). Mathematical Modeling: Can It Be Taught And Learnt? *Journal of Mathematical Modeling and Application*. Vol. 1, No. 1, 45-58.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modeling process. *ZDM, The International Journal on Mathematics Education*, 38(2), 86 – 95.

Borromeo Ferri, R. (2007). Modeling problems from a cognitive perspective. In: Haines, C. et al. (Eds), *Mathematical Modeling: Education, Engineering and Economics*. Chichester: Horwood, 260-270.

Carreira, S., Amado, N., & Lecoq, F. (2011). Mathematical Modeling of Daily Life in Adult Education: Focusing on the Notion of knowledge. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and Learning of Mathematical Modeling*, (ICTMA 14) (pp 199 – 210). New York: Springer.

Cobb, P., Yackel, E., & Wood, T. (1992). A Constructivist Alternative to the Representational View of Mind in Mathematics Education. *Journal for Research in Mathematics Education*, 23(1), 2 – 33.

Field, A. (2009). *Discovering Statistics Using SPSS*. London: Sage Publications Ltd.

Galbraith, P. & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modeling process. *ZDM, The International Journal on Mathematics Education,* 38(2), 143 – 162.

Girnat, B. & Eichler, A. (2011). Secondary Teachers' Beliefs on Modeling in Geometry and Stochastics. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and Learning of Mathematical Modeling,* (ICTMA 14) (pp 75 – 84). New York: Springer.

Greefrath, G. (2011). Using Technologies: New Possibilities of Teaching and learning Modeling – Overview. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and Learning of Mathematical Modeling,* (ICTMA 14) (pp 301 – 304). New York: Springer.

Geiger, V. (2011). Factors Affecting Teachers' Adoption of Innovative Practices with Technology and Mathematical Modeling. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and Learning of Mathematical Modeling,* (ICTMA 14) (pp 305 – 314). New York: Springer.

Haines, C.R. & Crouch, R., (2010). Remarks on modeling cycle and interpretation of behaviours. In R. A. Lesh, P. L. Galbraith, C.R. Haines, & A. Harford (Eds.), *Modeling Students' Mathematical Modeling Competencies*, (ICTMA13) (pp 145-154). New York, London: Springer.

Kaiser, G. (1995). Realitätsbezüge im Mathematikunterricht – Ein Überblick über die aktuelle und historische Diskussion. In G. Graumann, et al. (Eds.): *Materialen für einen realitätsbezogenen Mathematikunterricht* (pp 64 – 84). Bad Salzdetfurth: Franzbecker.

Kaiser, G. & Schwartz, B. (2006) Mathematical modeling as a bridge between school and university. *ZDM, The International Journal on Mathematics Education*, 38(2), 196 – 208.

Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modeling. *ZDM, The International Journal on Mathematics Education*, 38(2), 82 – 85.

Kaiser, C. & Maass, K. (2007). Modeling in lower secondary mathematics classroom – problems and opportunities. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds), *Modeling and Applications in Mathematics Education. The 14th ICMI Study,* 99 – 108. New York: Springer.

McAlleese, R. (1988). The Knowledge Arena as an Extension to the Concept Map: Reflection in Action, *Interactive Learning Environments*, 6(1), 1 – 22.

Novak, J.D. (1977). *A theory of Education*. Ithaca: Cornell University Press.

Novak, J.D. & Gowin, D.B. (1984). *Learning how to learn*. Cambridge: Cambridge University Press.

Novak, J. D. & A. J. Cañas, The Theory Underlying Concept Maps and How to Construct and Use Them, Technical Report IHMC CmapTools 2006-01 Rev 01-2008, Florida: Institute for Human and Machine Cognition, 2008, available at: http://cmap.ihmc.us/Publications/ResearchPapers/TheoryUnderlyingConceptMaps.pdf (last visited: March 29, 2011).

Orrison, M. & Yong, D. (2006). *Instructor's Guide to the Graph Complexity Project*; online on Yonghttp://mathforum.org/pcmi/hstp/sum2007/wg/supervisors/journal/instructorsguide.pdf (last visited: May 16, 2011).

Perrenet, J.C. & Adan, I. (2010). The Academic Merits of Modeling in Higher Mathematics Education; a Case Study**.** *Mathematics Education Research Journal*. Volume 22, Number 2, 121-140.Perrenet, J.C. & Adan, I. (2011). Fifteen Years of Experience with Modeling Courses in the

Eindhoven Program of Applied Mathematics. *Journal of Mathematical Modeling and Application,* Vol. 1, No. 4, 3-19.

Sowa, J.F. (1984). *Conceptual Structures: Information Processing in Mind and Machine*. Reading: Ma: Addison-Wesley.

Schoenfeld, A.H. (1985). *Mathematical Problem Solving*. New York: Academic Press.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on Mathematics Teaching and Learning*, 334 – 370. New York: MacMillan.

Spandaw, J. & Zwaneveld, B. (2009). Modeling in mathematics teachers' professional development. In V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello, (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education, Working group 11 Application and Modeling*. Lyon: Unversité 1 de Lyon. See www.inrp.fr/editions/cerme6.

Zwaneveld, B. (1999). *Kennisgrafen in het wiskundeonderwijs*. Maastricht: Shaker Publishing. [in Dutch].