

# TEACHING FOR ROBUST UNDERSTANDING OF ESSENTIAL MATHEMATICS

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Final Draft  
October 24, 2015

Paper presented at the Project IMPULS symposium, “Essential mathematics for the next generation: What to teach and how should we teach it,” Tokyo, Japan, October 12-19, 2015.

## Teaching for Robust Understanding of Essential Mathematics

Give a man a fish, and you feed him for a day; show him how to catch fish, and you feed him for a lifetime.<sup>1</sup>

We have been feeding children mathematical fish for hundreds if not thousands of years. It is time to teach them how to fish.

In what follows I lay out some ideas concerning the three arenas we were asked to discuss: (1) Curriculum (e.g. Standards, syllabus, the contents of mathematics we should teach), (2) Teaching (e.g. Productive teaching practices, incorporating technology for teaching), and (3) Teacher Education both in preservice and inservice. I will delve most deeply into a theoretical framework, and some pragmatics, related to items 2 and 3.

### *1. What mathematics should we teach? (And, is it possible to build curricula to do so?)*

I have been wrestling with this problem ever since I moved from mathematics to mathematics education. I was a practicing mathematician in 1974, when I read George Pólya's (1945) book *How to Solve It*. The book was a revelation. Pólya discussed the problem solving strategies (heuristics) used by mathematicians – among them “analogy,” “here is a problem related to yours and solved before,” “If you cannot solve the proposed problem,” reductio ad absurdum and indirect proof,” and “working backwards.”

It was clear that I used these strategies – and that I had not been explicitly taught them. (Another of Pólya's gems: a *method* is a trick that you have used twice, and remembered. Successful mathematicians have been trained by the discipline: as they build on their earlier successes, they develop the methods that produce their subsequent successes.) The challenge was, could students be taught such strategies? It turns out that I was in the right place at the right time – cognitive science was about to emerge as a discipline, and I could use the techniques of artificial intelligence to unpack aspects of human (rather than machine) problem solving. I began that work in earnest in 1975, and my book *Mathematical Problem Solving* was produced a decade later.

Thirty years later, there is a range of language to characterize my inquiries and the generalizations that spring from them. One might describe “problem solving strategies”; one might discuss “thinking mathematically” (the title of a superb 1982/2010 book by John Mason, Leone Burton, and Kaye Stacey); one might discuss “mathematical processes” as in NCTM's 1989/2000 volumes of *Standards*, “mathematical practices” as discussed in the 2010 Common Core Standards for Mathematics, or mathematical “habits of mind,” as advanced by Al Cuoco and Colleagues in 1996. To remain agnostic about

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<sup>1</sup> This quotation has been attributed to the bible; to the philosopher Maimonides; and it has been called an “ancient Chinese proverb.” Apparently the first traceable source is a story titled “Mrs. Drymond” by Anne Isabella Ritchie, the daughter of William Makepeace Thackeray. It was published in the American magazine *Littell's Living Age* in 1885, having appeared in an earlier undated issue of the British *Macmillan's Magazine*. See <<http://www.phrases.org.uk/meanings/give-a-man-a-fish.html>>.

language, I shall refer to this class of behaviors as *Productive Patterns of Mathematical Thinking, or PPMT*<sup>2</sup>.

To my mind, then, there are three things to consider: (A) content and (B) productive patterns of mathematical thinking (PPMT), and (C) curricula that might support them.

*A: Mathematical content.* I am somewhat agnostic with regard to content, in that I think curricular content can and should evolve in a natural way, as mathematics and its applications evolve. For example, we have seen an increased focus (in the U.S.) on statistical ideas, and on modeling and applications; I think these trends are salutary<sup>3</sup>. The central issue here is how students experience the content. Key desiderata highlighted by the authors of the Common Core State Standards are “focus” and “coherence;” I agree.

To those I would add “conceptually grounded” – or in more common language, that *mathematics should not appear as a set of arbitrary rules, but should make sense and be seen as such*. I am convinced that all of the formal mathematics studied in grades K-12 (at minimum) can be seen as the codification of patterns of sense making – and that if students have a set of experiences in which they make sense of various phenomena, then the mathematics that encapsulates what they have done will be seen as comfortable symbolic shortcuts rather than as arbitrary rules to be memorized<sup>4</sup>. If space permitted I would describe the way in which my students, reading the textbook, found the arc length formula “ $s = r\theta$ ” to be completely incomprehensible; but, having worked through a number of examples in class, came to intuit the formula for themselves.

Topic B, productive patterns of mathematical thinking, is more complex in two ways: (i) identifying the key ideas to be learned, and (ii) building the sequence of activities that would enable students to come to grips with them.

*B(i): identifying the most important productive patterns of mathematical thinking.* In my opinion we have not yet done a good job of identifying the “right” PPMT. Pólya gave us a good start with regard to problem solving, and my work helped to elaborate those ideas, but problem solving is only a subset of “mathematical thinking” – the goal in this case. The habits of mind described in Cuoco et al. (1996) are in the right spirit and of the right breadth, but they are not organized in ways that support coherent analytic decomposition or direct support for comprehensive curricular design. The processes in the 1989 NCTM *Standards* and 2000 *Principles and Standards* (problem solving, reasoning and proof, communication, connections, representation) are all important, but miss out on some aspects of mathematical thinking that were noted in the practices in the 2010 Common

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<sup>2</sup> The question of how to characterize “Content” versus “Productive Patterns of Mathematical Thinking” is vexing – and has been so for a very long time. In Appendix A I reproduce a relevant segment of my 1994 paper “What do we know about mathematics curricula?” Twenty years later, the substantial imbalance in the *Common Core State Standards* – three pages of Standards for mathematical practice out of 90 pages of standards (the rest devoted to content) indicates that, in the U.S., at least, the issue has not been successfully resolved, at the political or curricular level.

<sup>3</sup> In Japan, curriculum has been taken more seriously than in the US; it has been studied empirically and has evolved over time, on the basis of evidence. Such stability and evolution are very desirable traits.

<sup>4</sup> This has implications with regard to problem-based curricula: see (C) below.

Core State Standards: a drive toward certain forms of (appropriate) precision and clarity, and to look for and make use of structure. But the 3-page list in the Common Core is of widely disparate scope, with (as George Orwell would say) some practices more equal than others, and without adequate specification to be useful in curricular terms.

In sum, the field's understanding of productive patterns of mathematical thinking is not yet coherent or organized in such a way that it could provide solid underpinnings for curricular design.

I propose that the field should work to identify a small number of “big ideas” – fewer than a dozen PPMT, which could then be analyzed with the seriousness that has been devoted to content analyses, and fleshed out in ways that would support curricular development. The first part of this – identifying key PPMT – could be achieved with a few invitational conferences that brought together mathematicians and mathematics educators. The prospect of engaging in this work, not simply at the level of armchair philosophy but at the level of proposing candidate PPMTs and fleshing them out with examples that span the curriculum, could be exciting.

*B(ii): building the sequence of activities that would enable students to come to grips with the PPMT.* One of the main findings of my problem solving work (see esp. Schoenfeld, 1985) is that it is possible to analyze and teach complex problem solving strategies. There were two main aspects to this work. The first was the recognition that the strategies described by Pólya were not, in fact, coherent strategies: many of the processes that Pólya called strategies (e.g., “examine special cases,” or “consider an easier related problem”) were *families of strategies*. That is, there are perhaps a dozen different ways to examine special cases, or to consider easier related problems. The mathematician who was trained by the discipline could, in hindsight, see that he or she was using those strategies; but the student, when given the name of the strategy, would not know where to start. Hence, the first part of the work in making problem solving strategies accessible to students consisted of doing the necessary analyses: given the name of a family of strategies, to identify the strategies that comprised that family. (For example, depending on context, one can “exploit an easier related problem” by considering an analogous challenge that uses simpler geometric figures; by looking at an analogous challenge with a smaller number of variables; by looking at linear, quadratic, and cubic functions instead of arbitrary polynomials, etc.)

So, a first, analytic task would be to take the PPMT – which, I am sure, will at first manifest themselves as *families* of productive thinking patterns – and decompose them into coherent sub-families that can be recognized and taught. The second task is to do the empirical work necessary to find out what kinds of experiences would enable students to come to grips with those PPMT. We do have a precedent for this: this was precisely the work I did, for problem solving strategies at the high school and beginning college level. Chapter 6 of *Mathematical Problem Solving* (Schoenfeld, 1985) provided some empirical evidence that students could learn such strategies; the design of my problem solving courses (and their success, as documented in the book as a whole) indicated that it is possible to provide students with a sequence of experiences that supports them in

developing such understandings. I stress that this would be a very large undertaking. And that leads to point C.

*C: Curricula to support the learning of important content and productive patterns of mathematical thinking.* The question is, can we build usable curricula that support students in developing the kinds of PPMT we would like them to develop? Here, I am happy to say, the world has changed a great deal since my problem solving work and my 1994 paper “What do we know about mathematics curricula?”, in which I reflected on the same issues I discuss here.

My problem solving courses are “problem-based.” That is, the “curriculum” (materials to help students become effective problem solvers) consisted of a series of carefully selected problems, along with a pedagogy designed to introduce students to central aspects of mathematical thinking. In considering the issue of whether it would be possible to fashion a curriculum along those lines, there are three major questions:

1. Is it possible to cover “content” as well as productive patterns of mathematical thinking in a problem-based curriculum?
2. If students spend a large amount of time focusing on PPMT, will that result in a decrease in their understanding of core content?
3. Can aspects of the flexible and responsive pedagogy that such an approach demands be captured in curricular materials?

Those were very much open questions twenty years ago. Today there is a solid body of evidence to suggest that the answer to questions 1 and 3 is “yes,” and that the answer to question 2 is “no.” The primary evidence is the collection of 100 “Formative Assessment Lessons” or FALs produced by the Mathematics Assessment Project (2015), which are available at the project web site, <<http://map.mathshell.org/>>. The lead designer for the project is Malcolm Swan. I am assuming that his paper at this conference will say a great deal about the FALs, so I will keep my comments brief.

The 100 Formative Assessment Lessons (20 FALs per grade, at grades 6 through 10) are divided into “concept” and “problem solving” lessons, each of which require 2-3 days of instruction. They provide a clear existence proof that problem-based curricular units can present both content and productive patterns of mathematical thinking to students. This implies a positive answer to question 1.<sup>5</sup> Two independently commissioned studies provide rather compelling evidence with regard to questions 2 and 3. Herman et al. (2014) examined the impact of the FALs as part of the Gates Foundation’s Mathematics Design Collaborative (MDC) efforts across the state of Kentucky. (The study also evaluated to the Literacy design collaborative (LDC), which is referred to in what follows.) Herman et al. (2014) found the following:

Statistically significant results on students’ state assessment scores ... are results worth celebrating. These positive findings are particularly so in light of both study teachers’ limited prior experience implementing the tools and the limited dosage

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<sup>5</sup> I say “implies” because 20 FALS per year entail perhaps 50 days of instruction. It is still a long way to full curriculum coverage.

students experienced. That is, study teachers had only one or two years of experience with LDC or MDC prior to the study year, and for the great majority it was only one year. Based on research on teachers' implementation of new practices, this is insufficient time for teachers to become fully comfortable and competent with the kinds of new pedagogical practices that LDC and MDC represent (Coburn, 2003; Hargreaves & Fullan, 2012). ... MDC, for its part, emphasizes student-centered pedagogies and the engagement of students' mathematical thinking and problem solving learning, in contrast to the teacher-directed approaches and low-level learning goals that have been dominant in traditional mathematics curriculum (U.S. Department of Education, 2008).

Intervention dosage is another factor to consider. ... In general, the longer and more intensive the treatment, the more likely an intervention is to show measurable effects. ... For MDC, participating teachers were expected to implement between four and six Challenges<sup>6</sup>, meaning that students were engaged only 8-12 days of the school year. Nonetheless, the studies found statistically significant learning effects for both tools, the approximate equivalent of ... 4.6 months for MDC." (Herman et al., p. 10)

That is: in comparison to control groups, between 8-12 days of instruction using the FALs resulted in knowledge growth equivalent to 4.6 months of regular instruction. That is rather dramatic evidence of the "yes" answer to question 2. And it raises the question, how could such an intervention have so large an effect?

The answer, I believe, is that the FALs not only supported a different (more responsive, more student-centered) pedagogy, but that some of the pedagogical techniques teachers picked up by learning to teach the FALs wound up being incorporated into their regular practice. A second study (Research for Action, 2015) provides some substantiation for this perspective. Here is a summary statement from the report:

### **Teacher Practice**

In the spring of 2013, three years after the introduction of MDC, over 700 participating teachers shared their experiences in a survey. A large majority of teachers responding to the survey reported that MDC had positively affected their practice in multiple ways, including helping them to:

- Implement the Common Core State Standards;
- Take on the role of facilitator-coach;
- Raise their expectations for students;
- Learn new strategies for teaching subject matter;
- Use formative assessment; and
- Differentiate instruction. (Research for Action, 2015, page 2)

In short, there is strongly suggestive evidence that, if we were to do the kinds of analyses of productive patterns of mathematical thinking suggested in part B, we could (with very

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<sup>6</sup> The FALs are also referred to as "Classroom Challenges."

significant effort) build curricula that would support teachers in introducing students to mathematically important content *and* powerful modes of mathematical thinking.

*Postscript, written after our week of intense collaboration:*

There was a very strong consensus among all of the participants in the workshop that led to the Project IMPULS symposium that, in the various nations represented (at minimum, Japan, the US, and the U.K.) we have a good handle on issues related to content, but that much more attention needs to be given to elaborating the PPMT and learning how to help students learn them. There was also strong agreement with the idea advanced by Malcolm Swan that the range of products we ask students to produce in classrooms (typically, numerical answers to problems or arguments to confirm claims that students are told are true) should be expanded significantly, to include a range of performances for technical mastery, tasks that involve classification, representation, analysis, mathematical argumentation, modeling, and critiquing others' arguments.

## ***2. What are the most important dimensions of rich instructional environments?***

The main focus of my work in recent years has been to address the question, “What really matters in mathematics classrooms?” By that I mean,

*(\*\*) Is it possible to identify a small number of well defined dimensions of classroom practice, with the property that, classrooms that do well on those dimensions produce students who are really good mathematical thinkers and problem solvers? And, is it possible to support teachers in becoming increasingly good along those dimensions?*

The first part of (\*\*) – are there a small number of dimensions that matter? – is a research question, but one that has profound implications for teacher preparation and professional development. If, for example, there were dozens of essential classroom concerns, and no straightforward way to organize them, it would be extraordinarily hard to plan for effective instruction, or to refine one's practice. Where would one begin? Moreover, saying that a small number of things “count” is not enough: those things have to be “improvable.” That is, not only must they be identifiable in practice, but it should be possible to examine one's practice and find ways to get better at them<sup>7</sup>.

For reasons of space, I will not provide extensive detail on the origins of the origins of our framework, known as **Teaching for Robust Understanding of Mathematics**, or **TRU Math**. Details on the evolution of the framework, its rationale, and potential applications can be found in Schoenfeld (2013, 2014, 2015). A large number of supporting documents

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<sup>7</sup> This may seem obvious today, but I am old enough to remember when it was not. For example, through the 1970s a common research technique was to use factor analyses to cluster dozens of variables into a smaller number of factors that “mattered.” In the study of mathematical problem solving, “verbal ability” and “quantitative ability” were named as important factors. But, what did they mean, and how could they be improved? There were no good answers to these questions. Thus, that approach was a dead end. The point is that an analytic framework is only useful in practice if it is “actionable” – if the findings can be used to improve practice.

can be found the Algebra Teaching Study (2015) website, <http://ats.berkeley.edu/>, and the Mathematics Assessment Project (2015) website, <http://map.mathshell.org/>.

To summarize in brief, we found that although there existed a large number of classroom analysis frameworks, none had the properties described in (\*\*). Some excellent frameworks focused on one or two essential dimensions of classroom practice; but they were not comprehensive. Some widely used frameworks were not mathematics-specific, which we felt was problematic – in our opinion a focus on mathematics is essential. Others frameworks were comprehensive, but they listed dozens of important aspects of classroom environments, making them hard to comprehend and apply. A list of the frameworks we examined – and tried – is given in Appendix B.

Given that no extant frameworks met our desiderata, we began to construct a framework of our own<sup>8</sup>. Schoenfeld (2013) describes the twists and turns of our R&D process. To put things simply, our wish for comprehensiveness led to a massively complex and unwieldy framework<sup>9</sup>. With all the detail in front of us, the question was how to organize it. Ultimately, looking for “equivalence classes” (clustering together classroom activities that shared common properties) resulted in a framework that could be organized into five major dimensions. Given that the framework resulted from the distillation of activities that represented the union of all of the observation frameworks we knew, it is reasonable to believe that all important classroom activities are captured in the framework. The five dimensions of the Teaching for Robust Understanding of Mathematics (TRUmath) framework are given in Figure 1.

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<sup>8</sup> In addition to the properties highlighted in (\*\*), we also wanted to be able to observe and code an hour’s worth of classroom activities (most likely, captured on video) in no more than, say, two hours – allowing for the use of the framework in coding large numbers of videos.

<sup>9</sup> My initial attempts to build an observation framework were based on my analytic work on teachers’ decision making. They were extremely detailed, and unworkable.



The Five Dimensions of Mathematically Powerful Classrooms				
The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Formative Assessment
<i>The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.</i>	<i>The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.</i>	<i>The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the "air time" is not equitable.</i>	<i>The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority (recognition for being mathematically solid), resulting in positive identities as doers of mathematics.</i>	<i>The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to move forward.</i>

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Figure 1. The TRUmath framework: The five dimensions of powerful mathematics classrooms.

Before proceeding, let me say a bit about the character of the framework and where more supporting detail can be found. TRU was developed under the aegis of two major projects, the Mathematics Assessment Project (see <http://map.mathshell.org/trumath.php>) and the Algebra Teaching Study (see <http://ats.berkeley.edu/>). There are a large number of documents available on those web sites, providing the research underpinnings of the framework, a scoring rubric to be used for research purposes, and some tools for use in professional development.

A major feature of the framework is that *it focuses on the mathematical experience in the classroom from the student's point of view*. If one is concerned with students' mathematical learning, then the appropriate framing is: what is the student's classroom experience with mathematics? To be sure, what the teacher says and does is centrally important. But what matters is how the student perceives (and reacts to) what takes place in the classroom that matters. The teacher's actions, even though central, are only part of the story.

As indicated by Figure 1, there are five main dimensions to the TRU framework. Dimension 1, *the mathematics*, has to do with the quality of the mathematics experienced by the student. It goes without saying that the more the curriculum or teacher provide meaningful access<sup>10</sup> to richer mathematics, the greater the opportunities students have to emerge from the classroom as powerful mathematical thinkers. In section 1 of this paper I tried to outline my sense of the mathematical riches – content and process – that students should experience.

Dimension 2, *cognitive demand*, concerns the degree to which the environment provides students with opportunities for sense making and for what has been called “productive struggle.” A solid body of literature (see, e.g., Henningsen & Stein, 1997; Stein, Engle, Smith, & Hughes, 2008; Stein, Grover, & Henningsen, 1996) documents the fact that, even when curricula offer possibilities for “stretching” students in reasonable ways, teachers in the U.S. tend to “scaffold away” some of the challenge, thus decreasing opportunities to engage meaningfully<sup>11</sup>.

Needless to say, finding the “right” level of challenge for students is difficult – no student is “average,” which means that a “one size fits all” approach to content is bound to be problematic. That is, effective curricula/instruction is designed in such a way that it can “meet the students where they are” and help them grow from there. In the lesson study lessons I have observed in Japan, student thinking is *anticipated*, on the basis of experience. Then, as students work (typically individually, sometimes in groups), the teacher takes stock of student work and provides some assistance. This phase of instruction, known as *kikanshido*, has been characterized as follows:

The Japanese word *kikanshido* literally means “in-between desk instruction.” During *kikanshido*, the classroom teacher observes how each student is solving the problem, consider in what sequence the various solution strategies may be shared and discussed, and provide appropriate support to individual students. During *kikanshido*, however, the teacher does not spend too much time with a single student as a major goal of this phase is for the teacher to know how all students are approaching the problem. (APEC Education Network, 2015)

The sequencing of solutions has the potential both to link different student ideas and to “bring along” students who enter the conversation at different points, during *neriage* (whole class discussion). There is a more than passing resemblance between these ideas and the *5 Practices for Orchestrating Productive Mathematics Discussions* (Anticipating, Monitoring, Selecting, Sequencing, and Connecting) discussed by Peg Smith and Mary K. Stein in their 2011 book by that title. And, those ideas are implemented in the 100 Formative Assessment Lessons produced by the Mathematics Assessment Project (<http://map.mathshell.org/>). This whole space of activities – attending to student thinking by finding out what students know (possibly by pre-assessments and by close observation of student work in class), and being prepared to make productive use of it – is the realm

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<sup>10</sup> What I mean by “meaningful access” will be fleshed out in the discussion of dimensions 2 through 5.

<sup>11</sup> In my experience, the best planned lesson study lessons in Japan are sensitive to this issue, as are the Formative Assessment Lessons I am about to discuss.

of formative assessment (see Black and Wiliam, 1998, for the canonical discussion of formative assessment.) *Formative Assessment* is dimension 5 of the TRU framework – discussed somewhat out of order because when it is done effectively, formative assessment results in the level of cognitive demand suitable for productive struggle (Dimension 2).

Dimension 3 pertains to equitable *access to the mathematical content*, and I want to stress how I use the term here. There can be many definitions of “powerful classrooms.” For example, one might describe a powerful teacher (or classroom) as one that produces some students who turn out to be exceptional mathematical thinkers. The mathematician R. L. Moore, father of the “Moore Method,” is known for having produced a number of superb Ph.D.s; he certainly qualifies as an extraordinary teacher. However, his method selected the best and trained them, in the same way that Olympic training programs identify and support exceptional talent. In the context of public education, I believe it is essential for all students to have access to powerful mathematics. Hence the question I ask when observing classrooms is, “does every student have the opportunity to engage with the core content in meaningful ways?”

Finally, Dimension 4 relates to issues of *Agency, Authority, and Identity*. The motivating question here is, “How do students come to view themselves as thinkers and doers of mathematics?” We have all (at least in English speaking countries) met people who, when told that we are mathematics educators, say “I’m just not a math person. I gave up on mathematics when I couldn’t understand why [pick one: (i) you invert and multiply to divide fractions; (ii) multiplying two negative numbers results in a positive number; (iii) the quadratic formula; or...].” This idea of “not being a math person” is part of that person’s mathematical identity. (See, e.g., Cobb, Gresalfi, & Hodge, 2009; Gresalfi & Cobb, 2006; Langer-Osuna, 2007; Martin, 2000.) The issue with regard to classroom environments (since it is not possible to observe the formation of identity, which takes place over time) is, *what opportunities does the classroom environment provide for students to come to see themselves as people who can do and explain mathematics?*<sup>12</sup>

### *The status of TRU*

The working hypothesis of the TRU framework is that classrooms that do well on the five dimensions highlighted in Figure 1 will produce students who are powerful thinkers and problem solvers. In that it reflects a distillation of the literature, TRU represents, at some level, the things known to be important factors in classrooms. Hence, it is most unlikely that one of TRU’s dimensions will be shown not to matter empirically (that is, the five dimensions are all “necessary” for powerful classrooms, in the mathematical sense); and, unless our understanding of what counts shifts dramatically in the coming years, they should be “sufficient” as well<sup>13</sup>. It should be noted that such a distillation necessarily foregrounds some things, and backgrounds others. For example, one might ask where the

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<sup>12</sup> A large part of the conceptualization of this dimension is grounded in the work of Randi Engle (e.g., Engle, 2000).

<sup>13</sup> Of course, proving that one has taken everything into account is impossible. However, as time goes on, such claims, if not disproved, become increasingly plausible.

issue of students' emotional vulnerability and safety appears in Figure 1. The necessity of attending to safety is implied, in the sense that an environment in which students do not feel safe is one in which students will have a very difficult time developing a sense of mathematical agency, authority, and identity (Dimension 4). But that is not immediately obvious. One way to think about TRU is that the dimensions characterized in Figure 1 span the space of productive classrooms – they might be considered “basis vectors” in that space. It would be entirely possible to choose an alternate set of basis vectors, which might place different aspects of classroom environments in the foreground and background.

At this point in its development TRU is under-specified, in the sense that it hardly provides a “road map” for determining what is more or less powerful and productive in each of the dimensions. To examine any particular classroom with regard to Dimension 1 (the mathematics), one needs a detailed sense of what mathematics is appropriate at that time for those students, how it fits into the curriculum, whether it is being experienced in a way that is conceptually rich, etc. In the U.S. we are very far from that: the Common Core State Standards exist, but there are no curricula that really line up with them. Perhaps the situation is better in Japan, where curricula evolve more slowly and carefully. There is less known about Dimensions 2 through 5. Each of those dimensions represents a major research area. The field has some understandings of the affordances of various classroom methods for cognitive demand (along with formative assessment as a tool), equitable access, and the development of agency, authority, and identity – but, there is much more that we don't know than we do. Hence the tools in TRU are good approximations, based on what the field knows to date<sup>14</sup>; but there is much more work to be done (preferably in a cumulative, data-driven manner). Finally, TRU has not undergone large-scale statistical validation. This is an unfortunate matter of timing: we were developing the ideas for TRU simultaneously with the conduct of the Measures of Effective Teaching Project (2012), so we did not benefit from the advantage of a large-scale (and expensive!) study to see how scores on TRU compared with scores on other frameworks, in correlating with mathematical performance. Until large-scale studies are done, we can only rely on plausibility arguments. The content is not *that* different from other frameworks, though it is, in the author's opinion, cleaner. One would expect, at minimum, comparable correlations to those obtained with the other frameworks.

Given the absence of large-scale data, let me address the face validity of the framework – domestically in the U.S., and then outside the U.S. Within the U.S., the framework is getting some notice, and some traction. It has been used as a basis for professional development in cities such as Chicago and New York, and has been adopted as the framework organizing professional development efforts in California districts (including Los Angeles, Oakland, and San Francisco) that serve more than 1 million students. The challenge, if course, it to turn the ideas into productive programs of professional development – see section 3. I am confident that TRU will resonate in primarily English-speaking cultures, based on the fact that (a) it was distilled from those literatures, and (b) it offers a small number of comprehensible, “actionable” foci. There is nothing “new” in

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<sup>14</sup> See the next section for more detail.

TRU, in the sense that it distills what is known; but it does so in ways that can be kept in mind (five dimensions, or questions, can be easily learned) and that can be worked on.

A much greater question is whether TRU will resonate in other cultures. There is no question (cf. Stigler & Hiebert, 1999) that teaching is a cultural phenomenon: different natures have different organizations of schools, systems of curriculum development, curricular organization, teacher preparation and professional development, and teaching “styles.” Hence it is an open question as to whether TRU would make much sense in Japan, China, France, Germany, or other nations. I have begun some collaborative projects, and preliminary signs are encouraging, but it is too early to tell. During the 2015 Project IMPULS lesson study intensive program (see <https://www.facebook.com/ProjectIMPULS>), I produced a number of lesson commentaries using the TRU perspective. These seemed to be useful – highlighting some things that were tacit in the lesson study process, and might profit from being made explicit. The effort revealed strongly shared values, and potential synergy. That may or may not be the case with other nations; I am seeking collaborations to explore the issue.

### ***3. How can we use the understandings in (1) and (2) to enhance teacher preparation and professional development?***

Before proceeding, I want to be clear about what TRU is and what it is not.

*First and foremost, TRU is a research-based perspective on what matters in classrooms. It offers a framework for inquiring into classroom practice, and a language for discussing the most important aspects of classroom practice. It suggests that if a classroom has certain attributes (rich mathematics; an appropriate level of cognitive demand, most likely supported by formative assessment; equitable access to the mathematics, and opportunities for all students to develop productive senses of agency, authority, and identity) that the students who emerge from that classroom will be powerful mathematical thinkers and problem solvers.*

*TRU is not prescriptive: it does not say how to achieve those goals. It does not specify any particular style of teaching or classroom activities, because there may well be different ways to achieve the same ends.*

The idea of TRU as a perspective is fundamentally important. When I plan lessons; when I teach; when I observe lessons and reflect on them, I view those lessons through the perspective outlined by TRU. I consider the mathematical opportunities made available to students; I consider whether classroom activities provide opportunities for sense making and how they can be (or were) tailored to meet individual student performance; I look for opportunities for all students to engage with the mathematics, and for opportunities for them to discuss and present mathematics, to develop a sense of agency and identity. In short, TRU frames the way I plan and reflect on instruction. This, I think, will be its great power.

Before delving into the TRU tools and their use in teacher preparation and professional development, I need to emphasize what should be obvious: issues of teacher preparation

and professional development (TP/PD) are very much shaped by national context and traditions. What that means is that any suggestion that X should be done by way of teacher preparation professional development without taking into account the national (or indeed, local) context is, a priori, ill-founded. That is where the non-prescriptive nature of TRU is a significant asset. It describes the properties of a “healthy” classroom, not how to construct it. By way of analogy, consider various measures of a nation’s health, whether economic or physical. The consumer price index and the infant mortality rate are indicators of status; they do not in themselves suggest mechanisms for improvement, which will depend on national or local context.

### *Some Tools*

As I mentioned above, TRU is a working hypothesis. In order to test that hypothesis, we need data; and to gather such data, we need ways of scoring classrooms according to their alignment with the TRU framework. Along those lines, we have created a TRU scoring rubric (Schoenfeld, Floden, & the Algebra Teaching Study and Mathematics Assessment Project, 2014) that, for different classroom activity structures (whole class, small group, student presentations, individual student work) identifies what might be considered “levels of richness” along each dimension. The summary rubric is given in Figure 2. The full rubric is available at the Algebra Teaching Study and Mathematics Assessment Project web sites.

	The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Formative Assessment
	<i>How accurate, coherent, and well justified is the mathematical content?</i>	<i>To what extent are students supported in grappling with and making sense of mathematical concepts?</i>	<i>To what extent does the teacher support access to the content of the lesson for all students?</i>	<i>To what extent are students the source of ideas and discussion of them? How are student contributions framed?</i>	<i>To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas when potentially valuable or address misunderstandings when they arise?</i>
1	Classroom activities are unfocused or skills-oriented, lacking opportunities for engagement in key practices such as reasoning and problem solving.	Classroom activities are structured so that students mostly apply memorized procedures and/or work routine exercises.	There is differential access to or participation in the mathematical content, and no apparent efforts to address this issue.	The teacher initiates conversations. Students' speech turns are short (one sentence or less), and constrained by what the teacher says or does.	Student reasoning is not actively surfaced or pursued. Teacher actions are limited to corrective feedback or encouragement.
2	Activities are primarily skills-oriented, with cursory connections between procedures, concepts and contexts (where appropriate) and minimal attention to key practices.	Classroom activities offer possibilities of conceptual richness or problem solving challenge, but teaching interactions tend to "scaffold away" the challenges, removing opportunities for productive struggle.	There is uneven access or participation but the teacher makes some efforts to provide mathematical access to a wide range of students.	Students have a chance to explain some of their thinking, but "the student proposes, the teacher disposes": in class discussions, student ideas are not explored or built upon.	The teacher refers to student thinking, perhaps even to common mistakes, but specific students' ideas are not built on (when potentially valuable) or used to address challenges (when problematic).
3	Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities for engagement in key practices.	The teacher's hints or scaffolds support students in productive struggle in building understandings and engaging in mathematical practices.	The teacher actively supports and to some degree achieves broad and meaningful mathematical participation; <b>OR</b> what appear to be established participation structures result in such engagement.	Students explain their ideas and reasoning. The teacher may ascribe ownership for students' ideas in exposition, <b>AND/OR</b> students respond to and build on each other's ideas.	The teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 2. The TRU Math Summary Rubric

As a research tool, the TRU rubric will allow us to pursue the question of how performance on TRU correlates with student performance on challenging mathematics tasks. Such quantification is necessary to pursue research along the lines of the Measures of Effective Teaching (2012) study. But, it is also worth noting that the sequence of scores in TRU, from 1 to 2 to 3, represents what might be considered a developmental trajectory into increasing classroom richness. As such, the rubric can be used to benchmark performance, and to chart growth.

More important in practical terms is the development of tools and processes for planning and reflection. As noted above, a key value of TRU is that it provides a frame for thinking about teaching, before and after instruction. One tool built for that purpose is the TRU Math Conversation Guide (Baldinger & Louie, 2014). In the conversation guide, each dimension is framed as an arena for inquiry. The lead questions for each dimension are given in Figure 3:

Dimension	Key Question
The Mathematics	How do mathematical ideas from this unit/course develop in this lesson/lesson sequence?
Cognitive demand	What opportunities do students have to make their own sense of mathematical ideas?
Access	Who does and does not participate in the mathematical work of the class, and how?
Agency, Authority, and Identity	What opportunities do students have to explain their own and respond to each other's mathematical ideas?
Formative Assessment	What do we know about each student's current mathematical thinking, and how can we build on it?

Figure 3. Lead questions for planning and reflection, for each of the five dimensions

Each lead question is then expanded into a family of questions that a teacher, teacher and coach, or professional learning community can use for purposes of planning and reflection. For example, the lead question in “cognitive demand” is, “What opportunities do students have to make their own sense of mathematical ideas? In the expansion of this question, teachers are invited to think about:

- What opportunities exist for students to struggle with mathematical ideas.
- How students' struggles may support their engagement with mathematical ideas.
- How the teacher responds to students' struggles and how these responses support students to engage without removing struggles.
- What resources (other students, the teacher, notes, texts, technology, manipulatives, various representations, etc.) are available for students to use when they encounter struggles.
- What resources students actually use and how they might be supported to make better use of resources.



- Which students get to engage deeply with important mathematical ideas.
- How future instruction could create opportunities for more students to engage more deeply with mathematical ideas.
- What community norms seem to be evolving around the value of struggle and mistakes. (Baldinger & Louie, 2014, p. 8)

To repeat, the idea is for the Conversation Guide and other TRU tools to support planning and reflection, individually and collectively. In the next sections I discuss the ways we are planning to make use of the tools in teacher preparation and in professional development – in our own particular context. Use in other contexts is open to question (and planned collaboration).

### *On Teacher Preparation*

At Berkeley, we have a two-year combined Masters degree plus Teaching Credential program called MACSME (Masters and Credential in Science and Mathematics Education). Last year we decided to make TRU the foundation of the teaching part of our program, and we are in the midst of working on how to do so. A challenge is that beginning teachers (especially those who have grown up in the U.S. context, where they have mostly listened to teacher presentations and worked on worksheets) tend not to have a very sophisticated view of classrooms. They tend not to know what to look at when they enter a classroom as an observer (they are almost certainly not oriented toward looking at student thinking as a main locus of activity), and they tend to be judgmental – “That’s not how I would do it.” TRU is complex; one cannot expect to present it to teachers-to-be and have them be able to use TRU tools with any sophistication. The challenge, then, is to help beginning teachers build understandings that will enable them to view classrooms in a way consistent with TRU, and come to use that perspective in a productive manner.

The second week of the current semester I gave a three-hour intensive workshop to MACSME students, in which I showed six rather different video clips and we discussed them. As we talked about what they saw in the first three videos, I wrote their observations (all of which fit under one dimension or another of TRU) on the board, and organized them – the result being the TRU framework. We then looked in greater depth at aspects of three other tapes, showing how three different classrooms provided students with different opportunities for the development of agency, authority, and identity. This “deep dive” into one dimension of the framework indicated how one could reflect on how classroom activities can provide students with very different affordances for the development of mathematical identities.

I am no more sanguine about these students’ ability, after our class, to *use* the tools and ideas in TRU than I would be about their ability to (say) be gourmet cooks, having just participated in a 3-hour introduction to fine food preparation. But, my hope is that they now have a sense of the key variables. The director of our teacher preparation can now refer to them, setting as a goal that by the end of the semester (perhaps the year) they will be looking at classrooms in that way, and be able to make good use of TRU tools. Indeed, one possibility is that the student teachers will, over the course of the semester, build

their own versions of the TRU rubric and conversation guide. This would happen as they come to reflect, with increasing sophistication, on lessons they observe and plan.

### *On Professional Development*

Here we have some plans in the U.S., with partner districts, and plans for collaboration with the Project IMPULS team.

As I mentioned above, the Mathematics Assessment Project has developed 100 formative Assessment Lessons, which are entirely consistent with the TRU perspective – in that, teaching a FAL in the way suggested by the lesson plan provides rich opportunities for students, along all five dimensions. Imagine a high school math department, in which the FALs are part of the curriculum. At the beginning of Year 1, members of the university team, mathematics coaches from the district, and the department chair introduce the TRU framework. The department chair indicates that she will be teaching a FAL the following week, and invites the department to examine and improve her lesson planning, using the TRU framework as a lens for thinking about what will take place. Then, as in lesson study, the lesson is taught, observed, and videotaped; and there is a follow-up discussion, using TRU as the framework for conversations about what took place<sup>15</sup>. At that point, the chair asks for a volunteer who will lead the planning and instruction process with another FAL. The department works through a number of such lessons over the course of that first year.

Entering into the second year, the department has some familiarity with TRU as a perspective and as a frame for supporting deep conversations about teaching. Thus, it is ready to engage in a more full-fledged version of lesson study, looking more deeply both at student thinking and at the pedagogy that supports it. The goal of our TRU-Lesson Study synthesis<sup>16</sup> is to understand and enrich classroom practices. The lesson study format and process are designed to support explorations in depth regarding the richness of the content and the development of student understandings. Our intention is to use the TRU perspective to help shape the research questions that we use in the lesson study process – in which case we hope to combine the strengths of the two approaches. My personal hope is to engage in a similar process with my project IMPULS colleagues, as a mechanism for deepening my own understanding of productive classroom practices.

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<sup>15</sup> As we saw in the Project IMPULS lesson study intensive, this is a very natural thing to do: given that the lesson study lessons we observed were all focused on student thinking, the TRU framework was a natural “fit.”

<sup>16</sup> The TRU-Lesson Study work is a partnership between the Oakland (California) Unified School District, the SERP (Strategic Education Research Partnership) Institute, Mills College, and U.C. Berkeley. It is supported by National Science Foundation grant 1503454, “TRUmath and Lesson Study: Supporting Fundamental and Sustainable Improvement in High School Mathematics Teaching.”

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## Appendix A

The first segment (pp. 55 ff.) of “The Question of Content” from Schoenfeld, 1994.  
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### THE QUESTION OF CONTENT

What do we want students to learn in mathematics courses, and how do we characterize what they have learned? I begin with a story that casts the issue in sharp relief. The story is true, with proper names replaced by variables because the specifics of people and place are irrelevant – and because the conversation described might well have taken place at dozens of universities around the country.

When I joined the mathematics department at University X some years ago I had a chat with the department chair, Professor Y, about my problem solving course. Y was truly supportive. He endorsed the idea of offering the course and helped me decide the level at which it should be offered. (Previous incarnations of the course had been given for student groups that ranged from first year liberal arts majors to advanced mathematics majors.) We decided the course should be offered as a lower division elective, so that it would be accessible both to potential mathematics majors and to non-majors who might want a dose of mathematical thinking above and beyond formal course requirements. After we had assigned it a course number, and right after Professor Y mentioned that the course might become an attractive option for mathematics majors, the following dialogue took place.

- Y: Of course we can’t give credit toward the major for taking the course.  
AHS: Why not?  
Y: Because you’re not teaching content.

I said that there was indeed content to the course: In solving the problems I assigned in the course discussions the students did a reasonable amount of elementary number theory, geometry, and so on. But Y didn’t buy the argument, because the content wasn’t identified and packaged in the standard ways; I didn’t cover specific theorems, bodies of knowledge, etc. (And, it is true that the students and I work through material in the problem solving course at a much slower pace than we would in a lecture class.) So I tried a different tack.

- AHS: I’ll tell you what. The goal of the course is simple: I want the students to be able to solve problems I haven’t explicitly taught them how to solve. The real test is my final exam, which is a collection of difficult problems that don’t look very much like the ones I discuss in the course. Here’s a bet. Suppose I have 20 kids in the class, just the ones who’ve enrolled for it. You can hand pick 20 junior and senior math majors and give them the exam. I’ll bet you my semester’s salary that my kids will outperform yours on the final.

Y: I'm sure you're right, but we still can't give credit toward the major.  
You're not teaching them content – *you're just teaching them to think.*  
[emphasis added]

... From Professor Y's perspective, the mathematical content of a course (or a lesson, or a whole curriculum) is the sum total of the topics covered.

... The danger in this kind of "content inventory" point of view comes from what it leaves out: the critically important point that *mathematical thinking* consists of a lot more than knowing facts, theorems, techniques, etc.

... In terms of overall perspective, the difference can be captured in a simple phrase: I would characterize the mathematics a person understands by describing what that person can *do* mathematically, rather than by an inventory of what the person "knows." That is, when confronted with both familiar and novel situations that call for producing or using mathematical ideas, can someone do so effectively? Note that this performance standard is the one that Professor Y lives by in his professional life, and the one that he uses to judge his colleagues. University mathematicians get promotion and tenure by producing new mathematics – i.e., by developing new or deeper understandings, seeing new connections, solving unsolved problems, etc. In or out of the university, mathematicians earn their keep by posing problems and solving them, either by developing new approaches or by recognizing that a new problem, looked at in the right way, is closely related to a familiar (and solvable) one. The key point here is that mathematicians have to *use* what they know; just knowing isn't enough. It is ironic that this is precisely the standard I proposed for the evaluation of my problem solving class – could my students outperform others, who had far more training? – and that he rejected, because it didn't fit with his notion of content.



## Appendix B

### Frameworks for Examining Classroom Activity

- Classroom Assessment Scoring System (CLAS: Pianta, La Paro, & Hamre, 2008)
- Framework for Teaching (FfT: Danielson, 2011)
- Instructional Quality Assessment, (IQA: Junker et al., 2004)
- Mathematical Quality of Instruction (MQI: University of Michigan, 2006)
- Performance Assessment for California Teachers (PACT: PACT Consortium, 2012)
- Protocol for Language Arts Teaching Observations (PLATO: Institute for Research on Policy Education and Practice, 2011)
- Systematic Classroom Analysis Notation (SCAN: Beebe, Burkhardt, & Caddy, 1980)
- UTeach Teacher Observation Protocol (Marder & Walkington, 2012)

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<sup>i</sup> The author gratefully acknowledges the Bill and Melinda Gates Foundation (Grant number OPP 1115160, “Networked Improvement Community to Support Common Core State Standards Implementation”) and the National Science Foundation (Grant number 1503454, “TRUmath and Lesson Study: Supporting Fundamental and Sustainable Improvement in High School Mathematics Teaching”) for funding that supported the development of this paper. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the funders.